Liquidity and Information in Order Driven Markets

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Abstract

How does informed trading affect liquidity in order driven markets, where traders can choose between market orders (demanding liquidity) and limit orders (providing liquidity)? In a dynamic model of order driven markets we find that informed trading overall helps liquidity: a higher share of informed traders (i) improves liquidity as proxied by the bid-ask spread and market resiliency, and (ii) has no effect on the price impact of orders. Compared to market orders, limit orders have a smaller price impact by a factor of about four. The model generates other testable implications, and suggests new measures of informed trading.

Key words: Limit order book, volatility, trading volume, waiting costs, slippage, implementation shortfall, information acquisition, stochastic game.

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1 Introduction

Market liquidity is a central concept in finance, in particular in relation with asset pricing.\(^1\) According to Bagehot (1971), illiquidity is caused by asymmetric information, via the actions of liquidity providers. The liquidity provider, or market maker, which Bagehot identifies as the “exchange specialist in the case of listed securities and the over-the-counter dealer in the case of unlisted securities,” sets prices and spreads so that on average he makes losses from traders who possess superior information, but compensates with gains from uninformed traders, who are motivated by liquidity needs or simply trade on noise. Thus, the stronger the asymmetric information between the informed traders and the market maker, the larger the bid-ask spread needs to be so that the market maker at least breaks even. A large theoretical literature has since made Bagehot’s intuition rigorous.\(^2\)

Following Bagehot (1971), most of the theoretical literature assumes that liquidity providers do not possess any superior fundamental information.\(^3\) More recent evidence, however, has called into question this assumption. One reason is that most financial exchanges around the world have become “order driven,” meaning that any investor (informed or not) can provide liquidity by posting orders in a limit order book.\(^4\) Moreover, empirical evidence shows that there is an important premium for liquidity provision in order driven markets, and that informed traders do indeed use limit orders extensively.\(^5\) Despite the evidence, the literature has been largely silent on the order choice problem of informed traders, and especially on the effect of this choice on market liquidity. The goal of the present paper is to fill this gap.

We consider the following set of questions. What is the optimal order choice of an informed trader? How does a larger fraction of informed traders affect liquidity? What is the information content of limit orders and market orders? How does the market

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\(^2\)See Kyle (1985), Glosten and Milgrom (1985), or O’Hara (1995) and the references within.

\(^3\)Notable exceptions are Chakravarty and Holden (1995), Kaniel and Liu (2002), and Goettler, Parlour, and Rajan (2009).

\(^4\)Nowadays, most equity and derivative exchanges are either pure order driven markets (Euronext, Helsinki, Hong Kong, Tokyo, Toronto); or hybrid markets, in which designated market makers must compete with a limit order book (NYSE, Nasdaq, London). See Jain (2005).

recover after a liquidity or information shock? Can we use the time series of market orders and limit orders to infer the fraction of informed trading?

To address these questions, we consider a dynamic model of an order driven market. Risk-neutral investors arrive randomly to the market and trade in one risky asset. The asset’s fundamental value is time varying, and information about it is costly to acquire and process. Informed investors learn the current value of the asset, and decide whether to buy or sell one unit of the asset, and whether to trade with a market order or a limit order. Limit orders can subsequently be modified or cancelled.

Our first main result describes the optimal order choice of the informed trader. This is essentially a threshold strategy: the informed trader (referred to in the paper as “she”) submits either a market order or a limit order, depending on the magnitude of her privately observed mispricing, which is the difference between the fundamental value (privately observed) and the efficient price (the public expectation of the fundamental value). An extreme mispricing causes the informed trader to submit a market order, while a moderate mispricing causes a limit order. This result formalizes an intuition present for instance in Harris (1998), Bloomfield, O’Hara, and Saar (2005), Hollifield, Miller, Sandås, and Slive (2006), Large (2009).

Our second main result is that limit orders have a price impact, and this impact is about one fourth the price impact of a market order. This follows from the optimal strategy of the informed trader. Indeed, suppose a buy limit order arrives. With positive probability, this order comes from an informed trader who has observed a mispricing which is positive (hence the buy order) and moderate (hence the limit order). Thus, a limit buy order increases by a positive amount the efficient price (along with the whole limit order book). Moreover, because market orders are caused by more extreme mispricing, the price impact of a buy market order is larger in magnitude (about 4 times larger in our model) than the price impact of a buy limit order. In line with this prediction, Hautsch and Huang (2012, p.515) find empirically that market orders have a permanent price impact of about four times larger than limit orders of comparable size.

Our third main result describes the equilibrium bid-ask spread, and identifies a new component of this spread: the slippage component. We define slippage as the tendency

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Because we are interested in long run liquidity effects, we assume that the asset value is not constant, but follows a random walk. Thus, prices do not eventually reveal all the private information. In Goettler, Parlour, and Rajan (2009), the fundamental value is also time varying, but follows a Poisson process.
of an informed trader’s estimated mispricing to decay over time.\footnote{Finance practitioners sometimes refer to “slippage” with a different meaning than in our paper. For instance, Investopedia refers to the slippage of a large (potentially uninformed) market order, for which each additional unit executes at a worse price—a phenomenon also known as walking the book. In contrast, in our model slippage applies only to limit orders submitted by informed traders, and it occurs even when limit orders are for just one unit.} Slippage is due to the future arrival of other informed traders who correct the mispricing by submitting their orders. Thus, slippage induces an endogenous waiting cost for the informed trader, called the slippage cost. Furthermore, the informed trader also suffers from an adverse selection cost, since at the time of order execution she is potentially less informed than the future informed traders.\footnote{This is because the informed trader acquires information only when she enters the market. If instead she continuously observes the fundamental value, the adverse selection component is zero, but the slippage cost is still positive, as competition with future informed traders gradually erodes her initial information advantage.} We call the decay cost the sum of the slippage cost and the adverse selection cost.

The decay cost generates a tradeoff between limit orders and market orders: by trading with a limit order, an informed trader gains half the bid-ask spread, but loses from the decay cost. By trading with a market order instead, the informed trader loses half the bid-ask spread, but pays no decay cost. At the threshold mispricing, the informed trader is indifferent between a market order and a limit order. Hence, the decay cost corresponding to this threshold value is equal to the equilibrium bid-ask spread. According to the decomposition of the decay cost into the slippage cost and the adverse selection cost, the bid-ask spread is the sum of a slippage component and an adverse selection component.

To our knowledge, the slippage component is new to the literature. Huang and Stoll (1997) decompose the bid-ask spread into order processing costs, adverse selection costs, and inventory holding costs. In our model, since we abstract from inventory issues and order processing costs, we recover the adverse selection component, but, in addition, we show that the dynamic nature of information produces a new component, caused by slippage.

In practice, slippage costs and adverse selection costs can be estimated if we have information regarding the evolution of limit orders submitted by informed traders. One such example could be a portfolio manager that has information (alpha), and decides on a trading strategy that involves submitting limit orders. According to Perold (1988), the trading strategy of the manager should minimize the implementation shortfall, which is the sum of execution costs (of transactions that actually execute) and opportunity
costs (of transactions that fail to execute). While execution costs can be estimated ex ante using liquidity measures such as the bid-ask spread and price impact, opportunity costs depend on assessing the evolution of prices when orders do not execute. Thus, our model can be used to estimate the opportunity costs for an informed portfolio manager.

Our fourth main result indicates how liquidity is affected by the fraction, or share of informed traders, henceforth called the informed share. Surprisingly, a larger informed share overall has a positive effect on liquidity. More precisely, a larger informed share has (i) a negative effect on bid-ask spreads; (ii) no effect on the price impact; and (iii) a strongly positive effect on market resiliency. Moreover, a larger informed share has a positive effect on market efficiency by reducing the efficient volatility. The latter is defined as the publicly inferred volatility of the fundamental value, hence its inverse is a measure of dynamic efficiency: when the efficient volatility is small, the public has precise information about the fundamental value.

To understand the positive effect of the informed share on the resiliency and the bid-ask spread, note that a larger informed share implies that the informed traders exert more pressure on prices to revert to the fundamental value. This explains the strong positive effect of the informed share on the market resiliency. Also, it explains the negative effect of the informed share on the efficient volatility: when there are more informed traders, the public eventually learns better about the fundamental value, and the efficient volatility decreases. But the bid-ask spread is equal to the decay cost corresponding to the threshold mispricing. When the efficient volatility is smaller, the decay cost is also smaller because the average mispricing tends to be smaller. Hence, a larger informed share generates a smaller bid-ask spread.

To understand the neutral effect of the informed share on market depth, suppose the informed share is small, and a buy market order arrives. There are two opposite effects at play. First, when the informed share is small, it is unlikely that the market order comes from an informed trader. This effect decreases the price impact. But, second, if the buy market order does come from an informed trader, she must have observed a fundamental value far above the efficient price; otherwise, knowing there is

\footnote{Several empirical papers document this positive relationship, although the interpretation offered is different from ours. Collin-Dufresne and Fos (2013) find that the bid-ask spread and realized price impact decrease in the presence of corporate insider trading. In our model, price impact is not affected by informed trading, but this might be due to the fact that we measure the instantaneous price impact, while they estimate the realized price impact, which might be affected by market resiliency when investors are strategic. Brennan and Subrahmanyam (1995) find that more analyst coverage for a security improves its realized price impact.}
little competition from other informed traders, she would have submitted a buy limit
order. This effect increases the price impact. The two effects exactly offset each other. 10

The results described thus far assume that the equilibrium is in steady-state, mean-
ing that the efficient volatility (as well as the bid-ask spread and the price impact) is
constant. In steady-state, the natural increase in uncertainty due to changes in the fun-
damental value is exactly offset by the new information contained in the order flow. Our
final set of results arise from the study of the behavior of the limit order book after an
uncertainty shock—for instance, after unclear public news results in a temporary spike
in efficient volatility.

After an uncertainty shock, the equilibrium converges to the steady-state. The speed
of convergence to steady-state is larger when there are more informed traders (market
resiliency is increasing in the informed share). The bid-ask spread, as well as the amount
of adverse selection (measured by the price impact coefficient), are both increasing in the
size the uncertainty shock, and in general they are an increasing function of the efficient
volatility. After the initial spike, both price impact and the bid-ask spread revert to
their steady-state values, at the same speed of convergence as the efficient volatility.
We thus obtain liquidity resiliency, which represents the tendency of bid-ask spread and
price impact to revert to their longtime (steady-state) values. Liquidity resiliency is
different from market resiliency, which represents the tendency of prices to revert to the
fundamental value after an uninformative shock.

We introduce a new measure, the market-to-limit ratio, which is the defined as
the probability the next order is a market order, divided by the probability that the
next order is a limit order. This number is equal to one in steady-state, but after an
uncertainty shock the market-to-limit ratio drops to levels significantly less than one, as
the increase in the bid-ask spread temporarily prompts the informed traders to submit
more limit orders. The connections among the market-to-limit ratio with the liquidity
measures and the efficient volatility, as well as the expected evolution of the equilibrium
towards the steady-state, produce new testable implications of the model.

Overall, our theoretical model produces a rich set of implications regarding the
connection between the activity of informed traders and the level of liquidity. We find
that informed traders have on aggregate a positive effect, by making the market more
efficient and, at the same time, more liquid. A welfare analysis also shows that a larger

10 This is proved rigorously in Proposition 1 and explained in the discussion that follows it.
number of informed traders (caused for instance by an exogenous decrease in information costs) increases aggregate trader welfare. Our model thus provides useful tools to analyze informed trading, and its connection with liquidity, prices, and welfare.

Our paper is part of a growing theoretical literature on price formation in order driven markets.\textsuperscript{11} Of central interest in this literature is how investors choose between demanding liquidity via market orders and supplying liquidity via limit orders.\textsuperscript{12} Several papers, such as Foucault, Kadan, and Kandel (2005), or Roșu (2009) study order choice by assuming that investors have exogenous waiting costs. One advantage of our model is that waiting costs arise endogenously in the case of an informed investor: these are the decay costs described above.

Goettler, Parlour, and Rajan (2009) is the first paper that solves a dynamic model of order driven markets with asymmetric information. The focus of their paper is however different than ours. While we are interested in the effect of informed trading on liquidity, Goettler, Parlour, and Rajan (2009) analyze the interplay between information acquisition, order choice and volatility. They find that under picking off risks—which are absent in our model—different volatility regimes affect traders’ order choice, and make the market act as a volatility multiplier. Moreover, there are two important modeling differences. First, in their model private information is short-lived, because the fundamental value is publicly revealed after several periods. This assumption reduces the effect of dynamic efficiency in their model, as informed traders cannot arrive more quickly to make the market more efficient. By contrast, in our model dynamic efficiency has a strong effect by having private information being incorporated over the long run, and as a result the informed traders have an overall positive effect on liquidity. Second, in their model traders do not continuously monitor the market, which creates stale limit orders and picking off risks.\textsuperscript{13} In our model, there are no stale orders since limit orders can be modified instantly.

The paper is organized as follows. Section 2 describes the model. Section 3 solves for the equilibrium limit order book and the equilibrium payoffs of traders. Section 4 describes the properties of the equilibrium, including the various dimensions of liquidity


\textsuperscript{13}Linnainmaa (2010) finds that limit orders are often stale in the presence of public news.
and information efficiency. Section 5 explores non-steady-state equilibria of the model, as well as other extensions of our benchmark model. Section 6 concludes. Proofs of the main results are in the Appendix. A companion Internet Appendix contains additional results and robustness checks.

2 Model

We consider a dynamic model of trading in a single asset. Time is continuous and traders arrive randomly to the market. After deciding whether to acquire private information regarding the fundamental value of the asset, traders can submit an order to buy or sell one unit of the asset. Traders also choose the price at which they are willing to transact. If an order does not execute, it can be subsequently modified or cancelled. Information can be difficult to process, in a sense that is made precise below. We now describe the model in more detail.

Trading and Prices. The market mechanism is order driven, meaning that a transaction takes place when a buy or sell order is executed against an order on the opposite side. Each order is a limit order, as it specifies a quantity and a price beyond which the trader is no longer willing to transact. The price can be any real number. Limit orders are subject to price priority: buy orders submitted at higher prices and sell orders submitted at lower prices have priority. Limit orders submitted at the same price are subject to time priority: the earlier order is executed first. If several orders arrive at the same time, a random order is assigned to them.\footnote{With Poisson arrivals, the probability of two or more traders arriving at the same time is zero.}

The limit order book is the collection of all outstanding limit orders (submitted but not yet executed or cancelled). In the book, limit orders form two queues, based on order priority: the ask queue on the sell side, and the bid queue on the buy side. The lowest price on the ask side is the ask price, or simply the ask. The highest price on the bid side is the bid price, or simply the bid. A marketable limit order is a buy limit order with a price above the ask, or a sell limit order below the bid. A marketable limit order is executed immediately and is henceforth called a market order.

Traders and Arrivals. Traders arrive to the market according to a Poisson process with parameter $\lambda$. Immediately after arrival, a trader chooses whether to (a) submit a market order, (b) submit a limit order, or (c) submit no order at all. Each order is for
one unit of the asset. After submission, a limit order can be either (i) modified, which means the limit price is changed—in which case time priority is lost, or (ii) cancelled. As soon as the order is executed or cancelled, or if no order is submitted, the trader exits the model.

Traders are risk-neutral but their utility also includes a private valuation component and a cost from waiting. Each trader has a type $(u,r)$, which consists of a private valuation $u$ for the asset and a waiting coefficient $r$. The private valuation $u$ can take three possible values, $\{-\bar{u},0,\bar{u}\}$, where $\bar{u} > 0$. A trader is a natural buyer if $u = \bar{u}$, a natural seller if $u = -\bar{u}$, or speculator if $u = 0$. At time $t$, the instantaneous utility of a trader with private valuation $u$ is

$$U_t = \begin{cases} v_t - p_t + u, & \text{if trader buys at } t, \\ p_t - v_t - u, & \text{if trader sells at } t, \\ 0, & \text{if trader’s order does not execute at } t, \end{cases}$$

(1)

where $v_t$ is the fundamental asset value at $t$, and $p_t$ is the transaction price at $t$. Traders incur a waiting cost of the form $r \times \tau$, where $\tau$ is the expected waiting time, and $r$ is a constant coefficient. The waiting coefficient $r$ can take two possible values, $\{0, \bar{r}\}$, where $\bar{r} > 0$. A trader is patient if $r = 0$, or impatient if $r = \bar{r}$.

To simplify presentation, we assume that (i) impatient natural buyers always submit a buy market order, (ii) impatient natural sellers always submit a sell market order, and (iii) impatient speculators do not submit any order. In Internet Appendix Section 1, we show that (i)-(iii) are equilibrium results if $\bar{u}$ and $\bar{r}$ are above certain thresholds. Since traders who submit no order exit the model immediately, we replace (iii) by the assumption that all speculators are patient.

Natural buyers and sellers (traders with valuation $\bar{u}$ or $-\bar{u}$) arrive randomly to the market according to an independent Poisson process with parameter $\lambda^u$. They are equally likely to have positive or negative private valuation, and equally likely to be patient or impatient. Patient speculators arrive randomly to the market according to an independent Poisson process with parameter $\lambda^i$. The total trading activity is $\lambda = \lambda^u + \lambda^i$.

**Information.** At any time $t$, the asset has a fundamental value $v_t$, also called

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$^{15}$The private valuation can arise from liquidity or hedging needs, or from bias regarding the asset (optimism or pessimism). The waiting cost can arise from trading horizon/deadlines, or from uncertainty regarding future order execution.
common value or full-information price. The asset value follows a diffusion process \( dv_t = \sigma_v dB_t \), where \( B_t \) is a standard Brownian motion, and the fundamental volatility parameter \( \sigma_v \) is a positive constant. Because traders arrive to the market according to a Poisson process, inter-arrival times are exponentially distributed with mean \( 1/\lambda \). For simplicity of notation, throughout the paper we work in event time rather than calendar time: if a trader arrives at \( t \), the next trader arrives at \( t+1 \). The discrete version of the fundamental value process in event time is \( v_{t+1} = v_t + \sigma \varepsilon_{t+1} \), where \( \varepsilon_{t+1} \) has a standard normal distribution \( N(0,1) \), and the inter-trade volatility parameter is \( \sigma = \sigma_v / \sqrt{\lambda} \).

By paying an information acquisition cost, a trader learns the fundamental value at the time of arrival. To simplify presentation, we assume that only the patient speculators acquire information; this is proved as an equilibrium result in Internet Appendix Section 2. In what follows, we refer to the patient speculators as informed traders, and to the natural buyers and sellers as uninformed traders.

All traders observe the history of the game. As the market is order driven, the history consists of the total order flow, i.e., submissions, executions, modifications, and cancellations. The evolution of the limit order book and the transaction prices are part of this public information. A trader’s type (private valuation and waiting coefficient) is private information for each trader. The fundamental value at the time of arrival is private information for each informed trader.

**Equilibrium Concept.** Our model represents a stochastic game, in which Nature moves by drawing randomly new traders at each time \( t = 0, 1, 2, ... \). After traders arrive and decide whether to become informed or not, they engage in a trading game and at each time maximize their expected utility given their information set. Even though the arrivals occur at discrete points in time, traders can later modify their orders at any time in between. The game is therefore set in continuous time, and we use the framework of Bergin and MacLeod (1993) in which traders can react instantly.

The equilibrium concept is Markov Perfect Equilibrium (MPE), as defined for instance in Fudenberg and Tirole (1991). As a refinement of the Perfect Bayesian Equilibrium (PBE) concept, a MPE is defined by a game assessment, which is the collection of a strategy profile and a belief system such that (i) at every stage of the game, strategies

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16 This use of event time has been justified empirically for instance by Hasbrouck (1993). Equivalently, we can set the model in discrete time, in which case \( t + 1 \) is replaced by \( t + \frac{1}{\lambda} \).

17 Alternatively, the informed traders could continuously observe the asset value. This does not simplify the solution of the model, however, and we conjecture that the qualitative results are the same.
are optimal given the beliefs, and the beliefs are obtained from equilibrium strategies and observed actions using Bayes’ rule, and (ii) the game assessment is conditional on a set of state variables which are payoff-relevant. The latter condition implies that in a MPE there are no ad-hoc punishments to support the equilibrium.

**Information Processing.** Solving the model described above is very challenging if traders can do full Bayesian updating. This is because each trader’s inference problem involves an infinite number of state variables, which are the moments of the probability density ψ that describes the trader’s belief about the fundamental value. As new orders arrive, the belief ψ must be updated based on the information contained in each order type. But because informed traders use threshold strategies (see Theorem 1), the update of the density ψ changes its shape in ways which are difficult to quantify precisely.

Our modeling approach is to introduce frictions in information processing such that traders solve a simplified inference problem. These frictions are based on the principle that it is more difficult to process (i) private rather than public information, (ii) conditional rather than unconditional information, and (iii) higher rather than lower moments of a distribution. But rather than explicitly introducing information processing costs, we directly specify what information traders can process.

When updating the belief density ψ, an uninformed trader can compute without cost (i) the first moment of the posterior belief conditional on order flow, and (ii) the unconditional second moment of the posterior belief. Uninformed traders cannot compute higher moments, and their beliefs are always normally distributed. To avoid different beliefs among uninformed traders, we assume that the initial belief of an uninformed trader is such that after submitting a limit order in the direction of his private valuation, his posterior belief coincides with the posterior belief of the other uninformed traders. Thus, the uninformed traders waiting in the order book have the same normally distributed belief ψ^E, the *efficient density*. The efficient density is public knowledge. Its mean is the *efficient price* v^E, and its standard deviation is the *efficient volatility* σ^E.

Private information is more difficult to process. An informed trader cannot update the belief density ψ conditional on the order flow. But she can compute without cost the

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18 Given the difficulty of the traders’ inference problem and the fact that information acquisition is already costly in our model, it is plausible to assume that information processing is costly as well.
19 This assumption reconciles the divergence in beliefs that private knowledge about own type can create. For instance, an uninformed trader who submits a limit order privately knows that his order is uninformed, but the other uninformed traders do not know and may update their beliefs. See the proof of Lemma A3 in the Appendix for a formal discussion.
first moment of the relevant payoff variable, which is the asset mispricing (the difference between the asset value and the efficient price) at the time when her order is executed, provided that she follows the strategy of an uninformed trader after she makes the initial order choice. Equivalently, the informed trader makes the following decisions: (i) at the time of arrival she chooses which order to submit, based on the expected asset mispricing at the time of order execution, and (ii) if her choice is a limit order, she uses an uninformed broker to later update the order until it is executed.

For tractability, we assume that an informed trader receives a small penalty \( \nu \) if after observing the fundamental value she chooses not to trade.\(^\text{20}\) This assumption is equivalent to the informed trader receiving a private benefit \( \nu \) if she submits an order to the market, which intuitively can arise from “learning by trading.” Because \( \nu \) indicates a commitment to trade by the informed investor, we call it the commitment parameter. In Section 5.3, we show that this assumption is necessary only if the number of informed traders is above a threshold.

**Robustness.** The model described thus far can be solved essentially in closed form. We therefore use it as a benchmark model to study the robustness of the equilibrium results. In Section 5 we study the effect of relaxing some of the assumptions that are made for tractability. We then verify that the equilibrium is not significantly affected by relaxing these assumptions.

### 2.1 Notation and Parameters

The exogenous parameters in the model are the fundamental volatility \( \sigma_v \), the trading activity \( \lambda \), the uninformed trading activity \( \lambda^u \), and the informed trading activity \( \lambda^i \), subject to the equality \( \lambda = \lambda^u + \lambda^i \). The inter-trade volatility is \( \sigma = \sigma_v / \sqrt{\lambda} \). The investor preference parameters are the private valuation parameter \( \bar{u} \), the impatience parameter \( \bar{r} \), and the commitment parameter \( \nu \).

Let \( \phi(\cdot; M, S) \) be the normal density with mean \( M \) and standard deviation \( S \). Let \( \phi(\cdot) = \phi(\cdot; 0, 1) \) be the standard normal density, and \( \Phi(\cdot) \) its cumulative density. Let \( 1_X \) be the indicator function which equals one if \( X \) is true and zero if \( X \) is false.

We now define a set of numeric parameters that are used extensively throughout the text.

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\(^{20}\) This assumption is needed to avoid no-trade regions for the informed trader, which can occur when her perceived mispricing is close to zero.
paper. Let
\[ \alpha = \Phi^{-1}(3/4), \quad \beta = \frac{1}{4\phi(\alpha)}, \quad \gamma = \frac{\phi(0) - \phi(\alpha)}{\phi(\alpha)}. \] (2)
We have the following approximations: \( \alpha \approx 0.6745, \beta \approx 0.7867, \) and \( \gamma \approx 0.2554. \)

The fourth numeric parameter is the function \( g = g(\rho, w) \) from Definition 1 below. The function \( g \), as well as its arguments \( \rho \) and \( w \), has an interpretation within our model (see the discussion after Definition 1), but the definition itself is completely independent of this interpretation, and therefore \( g \) can be thought as a parameter. In particular, when we say in Definition 1 that the “order” \( O \) belongs to the set \( \{ \text{BMO, BLO, SLO, SMO} \} \), we simply mean that the element \( O \) belongs to a set with four elements that could be replaced for instance by \( \{1, 2, 3, 4\} \). In our model, however, these four elements are interpreted as the four types of orders that traders in our model can submit: buy market order (BMO), buy limit order (BLO), sell limit order (SLO), or sell market order (SMO).

**Definition 1.** Let \( \rho > 0 \). For every order \( O \in \{ \text{BMO, BLO, SLO, SMO} \} \), define
\[ \delta_O = \left\{ \frac{\alpha}{\beta}, \gamma \frac{\alpha}{\beta}, -\gamma \frac{\alpha}{\beta}, -\frac{\alpha}{\beta} \right\}, \]
\[ i_O = \{(\alpha, \infty), (0, \alpha), (-\alpha, 0), (-\infty, -\alpha)\}, \] and \( j_O = \{0, +1, 0, -1\} \).

If \( \psi \) is a density over \( \mathbb{R} \) and \( \Psi_O = \int_{z \in i_O} \psi(z)dz \), define \( \pi_{\psi, O} \) and the density \( f_{\psi, O} \) by
\[ \pi_{\psi, O} = \frac{1-\rho}{4} + \rho \Psi_O, \quad f_{\psi, O}(x) = \frac{1}{\pi_{\psi, O}} \int \left( \frac{1-\rho}{4} + \rho \mathbf{1}_{z \in i_O} \psi(z) \phi \left( x; z - \delta_O, \rho \sqrt{\frac{1+\gamma^2}{2\beta^2}} \right) \right) dz. \] (3)

Let \( j_0 \geq 1 \) be an integer, and \( Q = (O_0, O_1, \ldots, O_T) \) a sequence of orders, with \( T \geq 1 \). We say that \( Q \) is an “execution sequence” if \( j_0 + \sum_{t=1}^{T} j_{O_t} = 0 \) but \( j_0 + \sum_{t=1}^{T'} j_{O_t} \neq 0 \) for any \( 0 \leq T' < T \). Then, to every execution sequence \( Q = (O_0, O_1, \ldots, O_T) \) and density \( \psi_1 \) over \( \mathbb{R} \), we associate \( P(Q) = \prod_{t=1}^{T} P_t \) and \( \mu(Q) = \mu_{T+1} - \frac{\rho}{\beta} \), where we recursively define \( P_t = \pi_{\psi_{t, O_t}}, \psi_{t+1} = f_{\psi_t, O_t}, \) and \( \mu_{t+1} = E(\psi_{t+1}), \) for \( t = 1, \ldots, T \).

Let \( \rho \in (0, 1) \) and \( w \in \mathbb{R} \). Let \( j_0 = 1 \), and \( \psi_1 = \phi \left( \cdot; w - \gamma^2, \rho \sqrt{\frac{1+\gamma^2}{2\beta^2}} \right) \). Then the “information function” is
\[ g(\rho, w) = \sum_{Q \in \mathcal{Q}} P(Q)\mu(Q), \] (4)
where \( \mathcal{Q} \) is the set of all execution sequences of the form \( Q = (\text{BLO}, O_1, \ldots, O_T) \).

We are not able to find a closed form expression for the information function \( g(\rho, w) \). However, \( g \) can be estimated with good precision by using a numerical Monte Carlo procedure described in detail in Internet Appendix Section 3.
We introduce several more useful parameters:

\[
\begin{align*}
\rho &= \frac{\lambda^i}{\lambda^i + \lambda^u} = \text{informed share}, \\
\Delta &= \sqrt{\frac{2}{1+\gamma}} \frac{\sigma_v}{\sqrt{\lambda}} = \text{price impact parameter}, \\
\sigma_E &= \beta \rho^{-1} \Delta, = \text{efficient volatility parameter}, \\
s &= (\alpha - g(\rho, \alpha)) \sigma_E = \text{bid-ask spread parameter}.
\end{align*}
\tag{5}
\]

We now briefly explain how the information function \(g\) in Definition 1 is interpreted in our model, and introduce more useful notation. Recall that at any given date \(t\), the uninformed traders regard the asset value \(v_t\) as distributed by the normal density \(\psi_{E}^t\) (the efficient density), with mean \(v_{E}^t\) (the efficient price) and volatility \(\sigma_{E}^t\) (the efficient volatility). In the equilibrium described in Section 3, the efficient volatility is constant and equal to the parameter \(\sigma_E\) from (5). It is therefore convenient to normalize variables by \(\sigma_E\). For instance, we define the signal at \(t\) to be the normalized mispricing

\[
w_t = \frac{v_t - v_{E}^t}{\sigma_E}.
\tag{6}
\]

Then for the uninformed traders the distribution of the signal in the equilibrium of Section 3 is always standard normal, i.e., \(w_t \sim \mathcal{N}(0, 1)\).

Consider an informed trader who, for simplicity, arrives at date \(t = 0\) and observes an asset value \(v_0\) when the efficient price is \(v_{E}^0\). Then, \(g\) is a function of the informed share \(\rho = \frac{\lambda^i}{\lambda^i + \lambda^u}\) and the initial signal \(w_0 = \frac{v_0 - v_{E}^0}{\sigma_E}\). Suppose the informed trader submits a buy limit order (BLO), after which she follows the strategy of an uninformed trader (which is described in Corollary 4 below). Denote by \(T\) the random time when the BLO is executed (by an SMO). Then, \(g(\rho, w_0)\) is the expected signal \(w_T\) conditional on the BLO being executed at \(T\) by an SMO. Proposition 2 shows that this interpretation of \(g\) is indeed correct.

3 Equilibrium

In this section, we show that there exists a Markov Perfect Equilibrium (MPE) of the model. We also analyze the optimal strategies of the informed and uninformed traders, and the resulting expected utility. We then study the resulting equilibrium limit order
book and its evolution in time.

As with any MPE, the traders’ strategies depend on a set of state variables. In our context, the public state variables are (i) the efficient density, given by its first two moments, the efficient price and the efficient volatility, and (ii) the limit order book, given by the ask and bid prices, and the ask and bid queues. The private state variable is the asset value, observed by each informed trader when arriving to the market.\(^{21}\)

The MPE in this section has the additional property that it is a *steady-state equilibrium*, which we define in our context to mean that the efficient volatility is constant. In Section 5, we study non-steady-state MPEs corresponding to different initial values for the efficient volatility, and we show that all this equilibria converges to the steady-state MPE of this section.

The main difficulty in solving for the equilibrium is the inference problem of the informed trader. To understand why, consider an informed trader who arrives at date \(t\) and observes the asset value \(v_t\), or equivalently the signal \(w_t = \frac{v_t - v^E_t}{\sigma^E_t}\). Then, in order to decide what order to submit, she must be able to estimate for instance the payoff of a buy limit order (BLO). This is a complex problem, because she must take an average over all future order flow sequences that lead to the execution of her BLO. This payoff, however, can be computed easily if one knows the information function \(g(\rho, w)\) from Definition 1. We are only able to estimate \(g\) numerically, but conditional on knowing \(g\), the main formulas in the paper are given in closed form.

In the next result we verify numerically some properties of the information function.

**Result 1.** For all \(\rho \in (0, 1)\), the functions \(g(\rho, w)\), \(w - g(\rho, w)\) and \(g(\rho, w) - g(\rho, -w)\) are strictly increasing in \(w\), and

\[
\max\left(\frac{\rho (1 + \gamma)}{\beta}, -2g(\rho, 0) - \frac{2\rho \gamma}{\beta}\right) < \alpha - g(\rho, \alpha).
\]

Let \(g(\rho, w, j_0)\) be as in Definition 1 for any integer \(j_0 \geq 1\); \(g(\rho, \psi_1, j_0)\) for any density \(\psi_1\); and \(g_1(\rho, w, j_0)\) by taking \(\mu(Q) = 1\) in (4). Then, (i) \(g(\rho, \psi_1, j)\) increases if \(\psi_1\) has a positive shift in mean, (ii) \(g = g(\rho, w, j)\) decreases in \(j\) if \(w > 0\); and (iii) \(g_1 \equiv 1\).

Theorem 1 shows that there exists a MPE of the model if the conditions stated in Result 1 are satisfied. We verify these conditions numerically in Internet Appendix Section 3.

\(^{21}\text{More details are given in the proof of Theorem 1 in the Appendix.}\)
Theorem 1. Suppose the information function $g$ satisfies analytically the conditions from Result 1, and the investor preference parameters satisfy $\bar{u} \geq \frac{s}{2}$ and $\nu \geq \gamma \Delta$. Then, there exists a steady-state Markov Perfect Equilibrium of the game.

We describe the main properties of the equilibrium in the Corollaries 1–4 below. Corollary 1 describes the evolution of the efficient price, bid price, and ask price. Corollary 2 describes the initial order submission strategy of the informed trader. Corollary 3 shows that all types of orders are equally likely. Corollary 4 describes the initial strategy of the uninformed traders, and the subsequent equilibrium behavior of all types of traders in the limit order book.

Corollary 1. In equilibrium, the efficient volatility and the bid-ask spread are constant and equal, respectively, to the parameters $\sigma_E$ and $s$ from (5). If the efficient price is $v_t^E$, the ask price is $v_t^E + s/2$, while the bid price is $v_t^E - s/2$.

The efficient price changes only when a new order arrives. Let $\gamma \approx 0.2554$ be as in equation (2). If an order arrives at $t$, the efficient price changes from $v_t^E$ to (i) $v_t^E + \Delta$ if the order is BMO, (ii) $v_t^E + \gamma \Delta$ if the order is BLO, (iii) $v_t^E - \gamma \Delta$ if the order is SLO, and (iv) $v_t^E - \Delta$ if the order is SMO.

The first part of Corollary 1, that the efficient volatility and the bid-ask spread are constant over time, follows from the fact that the equilibrium of Theorem 1 is steady-state. We discuss this issue after Corollary 3.

To get intuition for the second part of Corollary 1, recall that the efficient price is the expected asset value given the public information (which coincides with the information of the uninformed traders). The efficient price does not change unless a new order arrives to the market. A new order affects the efficient price because each type of order contains private information. To give an example, according to Corollary 2 below, an informed trader submits a BMO if she observes an extreme asset value, i.e., an asset value $v_t$ above $v_t^E + \alpha \sigma_E$, or equivalently a private signal

$$w_t = \frac{v_t - v_t^E}{\sigma_E} \quad \text{(8)}$$

above $\alpha \approx 0.6745$. By contrast, an informed trader submits a BLO when the signal $w_t$ is positive but moderate, i.e., when the signal $w_t$ lies in the interval $(0, \alpha)$. This explains why a BMO shifts up the efficient price by $\Delta$, while a BLO shifts up the efficient price only by $\gamma \Delta \approx 0.2554 \Delta$. 

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Thus, the key to understanding the equilibrium is the strategy of the informed trader, which is described in the next result.

**Corollary 2.** Suppose an informed trader arrives at \( t \geq 0 \), and observes a signal \( w_t = \frac{v_t - v_{Ef}}{\sigma_E} \). Then, she submits a (i) BMO if \( w_t \in (\alpha, \infty) \), (ii) BLO if \( w_t \in (0, \alpha) \), (iii) SLO if \( w_t \in (-\alpha, 0) \), or (iv) SMO if \( w_t \in (-\infty, -\alpha) \). Her expected utility is, respectively,

\[
\begin{align*}
U^{I}_{\text{BLO}} &= \frac{s}{2} + \sigma_E g(\rho, w_t), \\
U^{I}_{\text{SLO}} &= \frac{s}{2} + \sigma_E g(\rho, -w_t), \\
U^{I}_{\text{BMO}} &= -\frac{s}{2} + \sigma_E w_t, \\
U^{I}_{\text{SMO}} &= -\frac{s}{2} - \sigma_E w_t.
\end{align*}
\]

(9)

To understand this result, suppose the informed trader gets a positive signal \( w_t \). Then, her main choice is between submitting a BMO and a BLO. By submitting a BMO, she gains from her signal \( (w_t) \), but loses half of the bid-ask spread \( (s/2) \) because she has to pay the ask price, which is higher than the efficient price by \( s/2 \) (see Corollary 1). By submitting a BLO instead, we see from equation (9) that the informed trader gains half of the bid-ask spread, and also benefits from her signal via the information function \( g(\rho, w) \).

The information function increases in \( w \) at a lower rate than \( w \) itself. Formally, this follows from Result 1, according to which \( w - g(\rho, w) \) is increasing in \( w \). Intuitively, this is because an informed trader who observes a large signal \( w_t \) knows that other informed traders are also likely to receive positive signals in the future, and therefore are more likely to submit buy orders. This bias towards buy orders therefore pushes up the efficient price in the future. In other words, the informed trader with a BLO expects to buy at a higher price in the future while she waits in the book. The stronger her signal, the stronger the bias, and therefore the stronger the relative penalty from submitting a BLO compared to a BMO. A more detailed discussion of this phenomenon, which is called the *slippage* of limit orders, is left for Section 4.

Because the function \( w - g(\rho, w) \) is increasing in \( w \), the payoff difference between BMO and BLO is increasing in \( w \). Therefore, for some threshold \( \alpha \), the informed trader prefers BMO for \( w_t > \alpha \), and BLO for \( w_t \in (0, \alpha) \). Intuitively, with an extreme signal the informed trader should use a market order, while with a moderate signal the informed trader should use a limit order. At the threshold \( w = \alpha \) (which occurs with zero probability), the informed trader is indifferent between BMO and BLO. The threshold \( \alpha \) is given by equation (2), and satisfies the property that for a variable \( w \) with
the standard normal distribution, the probability that $w > \alpha$ is equal to the probability that $w \in (0, \alpha)$ and is equal to 1/4.

**Figure 1: The Order Choice of the Informed Trader.** This figure displays (i) the efficient density, $\psi^E_t = \mathcal{N}(v^E_t, \sigma^2_E)$, which is the density of the asset value $v_t$ conditional on all public information until $t$, (ii) the four intervals on the horizontal axis that define the equilibrium order choice of an informed trader after observing $v_t$. The parameter $\alpha \approx 0.6745$ is as in equation (2).

Figure 1 illustrates the equilibrium order choice of the informed trader. The threshold between BMO and BLO is given by $w_t = \alpha$, or equivalently by $v_t = v^E_t + \alpha \sigma_E$. The normal curve in the figure represents the efficient density, which is the public belief about the asset value. The four regions under the curve and above the horizontal axis have area equal to 1/4 each, which reflects the fact that the informed trader submits each of the four order types with the same ex ante probability. Because the four types of orders are also equally likely for an uninformed trader, and because there are no cancellations in equilibrium, it follows that the four types of orders are equally likely in equilibrium given public information. We state this result in the next corollary.

**Corollary 3.** Conditional on public information, all order types (BMO, BLO, SLO, and SMO) are equally likely in equilibrium, with probability 1/4.

We call this equilibrium property *dynamic market clearing*. It is equivalent to the

---

22Indeed, the four types of uninformed traders arrive with equal probability, and the patient natural buyers submit BLO and the patient natural sellers submit SLO (see Corollary 4), while the impatient natural buyers submit BMO and the impatient natural sellers submit SMO.
following two properties: (i) buy and sell orders are equally likely, and (ii) market and limit orders are equally likely. It is the second property that is key for the intuition regarding dynamic market clearing. Suppose for instance that market orders were more likely than limit orders. Since every market order is executed against a limit order, the limit order book would become thinner over time, and therefore the equilibrium would not be steady-state. Thus, dynamic market clearing occurs because the equilibrium in Theorem 1 is steady-state. In Section 5.1, we analyze non-steady-state equilibria of the model, and we see that the dynamic market clearing condition no longer holds.

The next corollary describes the initial order submission decision of the uninformed traders, as well as their subsequent strategy if they submit a limit order. We only need to understand the patient uninformed traders, since the impatient ones always submit market orders. Also, because the informed traders are essentially uninformed after the initial order choice, the subsequent equilibrium behavior of the informed and uninformed traders coincides. In this sense, we say that the equilibrium is pooling.

**Corollary 4.** Consider a patient uninformed trader with private valuation \( u \) larger in absolute value than \( \Delta - s/2 \). Then, he submits a BLO if he is a natural buyer, and a SLO if he is a natural seller. In both cases his expected utility is

\[
U^U = \frac{s}{2} - \Delta + \bar{u}. \tag{10}
\]

After the initial limit order is submitted, the uninformed trader modifies his order along with the efficient price, as in Corollary 1. If an informed trader chooses to submit a limit order, her subsequent behavior mimicks the behavior of an uninformed trader. Traders in the limit order book modify their orders such that their relative ranks in the bid queue or the ask queue are always preserved. The equilibrium is pooling, in the sense that public information cannot be used to determine a trader’s type.

The part of Corollary 4 that refers to the uninformed traders is straightforward. Indeed, a patient natural buyer who submits a BLO gains half of the bid-ask spread \((s/2)\) and his private valuation \((\bar{u})\), but loses from the adverse selection of the SMO that eventually executes his order (from Corollary 1, the price impact of a SMO is \(-\Delta\)). Hence, as long as his private valuation is large enough to make his expected utility in (10) positive, he optimally submits a BLO. After submitting the initial order, the uninformed trader simply modifies his order according to the evolution of the efficient
price, because he is risk-neutral and updates his estimate of the asset value according to the efficient price.

Intuitively, the equilibrium is pooling, because traders do not have an incentive to deviate from the equilibrium. Indeed, if a trader were to jump ahead in the ask or bid queue, this out-of-equilibrium behavior would be interpreted immediately as coming from an informed trader with positive information. This information then would move the efficient price in such a way as to reduce the information advantage of the informed trader, while preserving ranks in the ask and bid queues. This reduction in expected payoff would then prevent the trader from deviating in the first place.

Figure 2: The Effect of Order Flow on the Limit Order Book. This figure displays the equilibrium shape of the limit order book just before trading at $t$ (middle plot), as well as the shape of the book at $t+1$ after a buy market order BMO (right plot) or a buy limit order BLO (left plot). For simplicity, the efficient price is set to $v^E_t = 0$, so that before trading at $t+1$, the efficient price becomes $v^E_{t+1} = \Delta$ after BMO, or $v^E_{t+1} = \gamma \Delta$ after BLO. The parameter $\gamma \approx 0.2554$ is as in equation (2).

Normally, without additional assumptions one should not expect the equilibrium
limit order book in our model to have a well defined shape. Indeed, trading is for only one unit, and without any modification cost the exact position of limit orders away from the bid and ask does not matter. However, the equilibrium shape of the limit order book can be fixed if we impose an infinitesimal cost of modifying limit orders. Suppose that when a limit order is executed at the ask, there is an infinitesimal modification cost for all the remaining limit orders on the ask side (and similarly for the bid side).

The resulting equilibrium limit order book is described in Figure 2. The middle plot in Figure 2 describes the typical shape of the limit order book just before trading at $t$. For simplicity, the efficient price is set at $v_t^E = 0$. The right and the left plots, respectively, describe the effect of a BMO or a BLO on the limit order book. To understand the assumption about the infinitesimal modification cost, suppose a BMO arrives at date $t$, when the limit order book is in the middle plot of Figure 2. Then, the SLO of trader $S_1$ is executed, and trader $S_2$ becomes the first in the ask queue. An instant later, $S_2$ should immediately modify his SLO at $v_t^E + s/2 + \Delta$, and therefore, with an infinitesimal modification cost, $S_2$ would prefer to have his order already at that price.

4 Market Quality and Informed Trading

In this section, we consider several measures of market quality and analyze how they are affected by the informed share, which is the fraction of order flow generated by the informed traders. As measures of market quality, we consider the information efficiency, as well as three measures of liquidity: the price impact, the bid-ask spread, and the market resiliency. In the process, we also study the information content of the different types of orders.

4.1 Information Efficiency

In general, a market is efficient at processing information if pricing errors are small. In our model, the pricing error is the difference between the fundamental value $v$ and the efficient price $v^E$, and the standard deviation of this pricing error is the efficient volatility. But in equilibrium the efficient volatility is constant (see Corollary 1), and it is equal to the parameter $\sigma_E$ from (5), which satisfies

$$\sigma_E = \beta \rho^{-1} \Delta,$$  (11)
where $\Delta = \sqrt{\frac{2}{1+\gamma^2}} \frac{\sigma_v}{\sqrt{\lambda}}$, and $\beta$ and $\gamma$ are constants defined in (2). Therefore, we propose the following measure of information efficiency:

$$\frac{1}{\sigma_E^2} = \frac{\rho^2}{\beta^2 \Delta^2}. \tag{12}$$

Indeed, when the market is informationally efficient, the efficient volatility is small, and therefore the proposed measure is large.

Because $\beta$ and $\Delta$ are independent of $\rho$, the information efficiency is increasing in the informed share $\rho$.\textsuperscript{23} It follows that information efficiency is increasing with the informed share. This shows that when there are more informed traders ($\rho$ is large), the order flow is more informative, hence the market is more efficient at processing information.

An interesting aspect of the increase in information efficiency is that it arises from the dynamic nature of the equilibrium. In a steady-state equilibrium, the opposite happens: when there are more informed traders the adverse selection is larger, and therefore the market is less informationally efficient. This intuition is discussed in more detail below, after Proposition 1.

The efficient volatility $\sigma_E$ can be used to estimate in practice the informed share. The problem is that it depends on other parameters of the model, such as the fundamental volatility $\sigma_v$ and the total trading activity $\lambda$. To remove this dependence, we consider the ratio of the inter-trade volatility ($\sigma = \sigma_v/\sqrt{\lambda}$) to the efficient volatility ($\sigma_E$),

$$\frac{\sigma}{\sigma_E} = \rho \sqrt{\frac{1+\gamma^2}{2\beta^2}} \approx 0.9277 \rho < 1. \tag{13}$$

The ratio $\sigma/\sigma_E$ provides a clean estimate of the informed share $\rho$, in the sense that the ratio does not depend on additional parameters. The inter-trade volatility $\sigma$ is in principle observable, as the price variance between trades. The efficient volatility is not observable directly, but it can be proxied by the dispersion of financial analysts’ estimates. Since (as we show in Section 4.3), the bid-ask spread $s$ is decreasing in the informed share $\rho$, a testable implication of equation (13) is that stocks with a lower ratio of inter-trade volatility to efficient volatility have larger bid-ask spreads.

\textsuperscript{23}The fact that $\Delta$ is independent of $\rho$ is obvious from its formula. The economic interpretation of this fact, however, is not obvious, and we discuss it in Proposition 1 and the paragraphs that follow it.
4.2 Price Impact

We define the price impact of an order as the effect of one additional unit of trading on the transaction price. Since all trades in our model are for one unit, our marginal price impact measure is the same as the effect of one unit on the efficient price. Because there are four types of orders, each order type $O \in \{BMO, BLO, SLO, SMO\}$ has a different price impact, which we denote by $\Delta_O$. Corollary 1 implies the following result.

**Proposition 1.** The price impact $\Delta_O$ of any order $O \in \{BMO, BLO, SLO, SMO\}$ is

$$
\Delta_{BMO} = \Delta, \quad \Delta_{SMO} = -\Delta, \quad \Delta_{BLO} = \gamma \Delta, \quad \Delta_{SLO} = -\gamma \Delta, \quad (14)
$$

where $\gamma \approx 0.2554$ is as in equation (2), and $\Delta = \sqrt{\frac{2}{1+\gamma^2}} \frac{\sigma_w}{\sqrt{\lambda}} \approx 1.3702 \frac{\sigma_w}{\sqrt{\lambda}}$ is as in equation (5). In particular, $\Delta_O$ does not depend on the informed share $\rho$. Moreover, the variance of the price impact is equal to the inter-trade variance $\sigma^2 = \frac{\sigma_w^2}{\lambda}$, that is,

$$
\text{Var}(\Delta_O) = \frac{1 + \gamma^2}{2} \Delta^2 = \frac{\sigma_w^2}{\lambda}. \quad (15)
$$

The reason why all order types have price impact is given by the usual adverse selection argument. Indeed, when setting the efficient price, the uninformed traders take into account the information contained in the order flow. For instance, if a BMO is submitted at $t$, then with positive probability it comes from an informed trader with a large signal $w_t = \frac{v_t - v^E_t}{\sigma^E_t} \in (\alpha, \infty)$. Then, the efficient price should increase (by $\Delta$). Similarly, if a BLO is submitted at $t$, then it might arrive from an informed trader with a moderate signal, $w_t \in (0, \alpha)$. Then, the efficient price should increase as well, although by a smaller amount (by $\gamma \Delta$).

A surprising implication of Proposition 1 is that the informed share $\rho$ has no effect on $\Delta$. To give intuition for this result, we note that there are two opposite effects of the informed share on $\Delta$. Suppose the informed share is small, and a buy market order arrives. The first effect is the usual adverse selection effect (see for instance Glosten and Milgrom (1985)): because $\rho$ is small, it is unlikely that the market order comes from an informed trader. This reduces the adverse selection coming from informed traders, and therefore decreases the price impact. But there is a second effect, the dynamic

\footnote{Alternatively, given the equilibrium shape of the limit order book (see Figure 2), we can also define the instantaneous price impact of a multi-unit market order, even though such orders are not part of the model. Then, as the size of the market order increases, each additional unit trades at a price changed by $\Delta$. This shows that the two definitions are consistent.}
**efficiency effect**: if the buy market order *does* come from an informed trader, she must have observed an asset value far above the efficient price; otherwise, knowing there is little competition from the other informed traders, she would have submitted a buy limit order. This effect increases the price impact.

Intuitively, the fact that the two effects exactly cancel each other follows from the equilibrium being steady-state. Indeed, in Internet Appendix Section 5, we show more generally that in a steady-state equilibrium the change in asset value and the change in efficient price must have the same variance. In our case, this translates to $\text{Var}(v_{t+1} - v_t) = \text{Var}(v^E_{t+1} - v^E_t)$. But the variance of the asset value change is the inter-trade variance $\sigma^2$, which does not depend on the informed share, while the variance of the efficient price change is $\text{Var}(\Delta_O)$, which according to Proposition 1 is a constant multiple of $\Delta^2$. Therefore, the price impact $\Delta$ is independent of the informed share $\rho$.

Proposition 1 yields a testable implication of our model, namely that the ratio of the price impact of a buy market order to the price impact of a buy limit order is

$$\frac{\Delta_{\text{BMO}}}{\Delta_{\text{BLO}}} = \frac{1}{\gamma} \approx 3.9152,$$

which is close to 4. Interestingly, Hautsch and Huang (2012, p.515) find empirically that market orders have a permanent price impact of about four times larger than limit orders of comparable size. In Section 5, we consider several extensions of the model, and study under what conditions and by what amount the price impact ratio is different from the benchmark value in (16).

### 4.3 Bid-Ask Spread

Another measure of liquidity is the bid-ask spread, which is by definition the difference between the ask price and the bid price. Corollary 1 implies the equilibrium bid-ask spread is constant and is equal to the parameter $s$ from equation (5).

**Corollary 5.** The equilibrium bid-ask spread is constant over time, and is equal to

$$s = (\alpha - g(\rho, \alpha)) \sigma_E.$$  

---

25Formally, when $\rho$ is small, the informed trader’s threshold for the choice between BMO and BLO is large. Indeed, Corollary 2 implies that the threshold signal is $w_t = \alpha$, or equivalently $v_t = v^E_t + \alpha \sigma_E$. But, as discussed in Section 4.1, the efficient volatility $\sigma_E$ is decreasing in $\rho$. 

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To get more intuition about the equilibrium bid-ask spread, we explain how the information function \( g \) from Definition 1 is interpreted in our model. Consider an informed trader who arrives at \( t = 0 \), observes a signal \( w = \frac{v_0 - v_E}{\sigma_E} \) and submits a BLO (which is not necessarily optimal). Assuming that subsequently all investors follow their equilibrium strategies, the informed trader then forms an expectation about the average asset value, based on all possible future order flow that executes her BLO at a later random time \( T > 0 \). The fact that the BLO is executed at \( T \) means that (i) the BLO is the first order in the bid queue before trading at \( T \), and (ii) a SMO is submitted at \( T \).

To state the next result, we introduce some notation. Let \( E_t \) be the informed trader’s expectation conditional on her information set before trading at \( t \), \( J_t = \{w, O_1, \ldots, O_{t-1}\} \), and let \( E^e \) be the informed trader’s expectation at \( t = 0 \) over all future execution sequences, i.e., over order sequences \( O_1, \ldots, O_T \) that execute the BLO at some \( T > 0 \).

**Proposition 2.** Consider an informed trader who observes at \( t = 0 \) a signal \( w \), and submits a BLO, which is executed at a random time \( T > 0 \). Let \( \rho \in (0, 1) \) be the informed share. Then, the information function \( g \) satisfies

\[
g(\rho, w) = E^e E_{T+1}(w_T).
\]

(18)

According to Proposition 2, \( g \) is the informed trader’s initial expectation of the signal at execution \( (w_T) \) conditional on the execution sequence, including the final SMO (hence the subscript “\( T + 1 \)” for the expectation in equation (18)). Then, the difference \( w - g(\rho, w) \) can be interpreted as the signal decay between the initial submission of the BLO until after its execution. It is therefore a cost that the informed trader faces when submitting a BLO (relative to submitting a BMO). We call \( (w - g(\rho, w)) \sigma_E \) the information decay cost, or simply the decay cost. Corollary 5 implies that the bid-ask spread \( s \) is precisely equal to the decay cost at the threshold signal \( w = \alpha \).

**Corollary 6.** Let \( \text{Decay Cost} = (w - g(\rho, w)) \sigma_E \) be the information decay cost faced by an informed trader. Then, the bid-ask spread \( s \) satisfies

\[
s = \text{Decay Cost}_\alpha.
\]

(19)

---

The expectation operator \( E^e \) is biased, because it is taken only on a subset of the future order flow sequences (the executable ones). As a result, the law of iterated expectations does not hold. As we explain below, this bias is caused by the phenomenon of “slippage.”
The intuition for this result is as follows. If the informed trader submits a BMO, she immediately captures her whole signal \( w \), but loses half of the bid-ask spread \( s/2 \). If she submits a BLO instead, she expects the future informed traders to increase the efficient price by also submitting buy orders, resulting in a decrease of her future signal. In other words, she expects that by the execution time \( T \) the signal \( w_T \) will decrease significantly. But this is exactly what the information function \( g \) measures. Thus, if the informed trader submits a BLO, she gains half of the bid-ask spread \( s/2 \), but captures only part of the signal \( g(\rho, w) \). Hence, the relative payoff difference between BMO and BLO is Decay Cost \( w - s \). Since at the threshold \( w = \alpha \) the informed trader is indifferent between BMO and BLO, it follows that the equilibrium bid-ask spread is equal to the information decay cost at the threshold.

To get more intuition about the bid-ask spread, we decompose it into two components. The first component, called the slippage component, corresponds to the informed trader’s information decay from the initial submission of the BLO until before its execution by the final SMO. The second component, called the adverse selection component, corresponds to the informed trader’s information decay due to the final SMO. To define these components, we introduce two functions similar to the information function \( g \).

**Definition 2.** For \( \rho \in (0,1) \) and \( w \in \mathbb{R} \), define the “slippage function” \( g^s(\rho, w) \) in the same way as the information function \( g(\rho, w) \) from Definition 1, except that the expression \( \mu(Q) = \mu_{T+1} - \frac{\theta}{h} \) is replaced with \( \mu(Q) = \mu_T \). Define the “adverse selection function” as the difference \( g^a = g - g^s \). Let

\[

g^s(\rho, w) = (\alpha - g^s(\rho, \alpha)) \sigma_E, \quad g^a = s - s^a = (g^s(\rho, \alpha) - g(\rho, \alpha)) \sigma_E.
\]

We call \( s^s \) the “slippage component,” and \( s^a \) the “adverse selection component.”

The next result provides interpretation of the functions \( g^s \) and \( g^a \).

**Proposition 3.** In the context of Proposition 2, the slippage function \( g^s \) and the adverse selection function \( g^a \) satisfy

\[
g^s(\rho, w) = \mathbb{E}^\sigma \mathbb{E}_T(w_T), \quad g^a(\rho, w) = \mathbb{E}^\sigma (\mathbb{E}_{T+1}(w_T) - \mathbb{E}_T(w_T)).
\]

Recall that the information function \( g \) can be interpreted as the informed trader’s initial expectation of the signal at execution \( (w_T) \) conditional on the execution sequence.
including the final SMO. By Proposition 3, the slippage function \( g^s \) is the same expectation, but conditional on the execution sequence without the final SMO. The difference is the adverse selection that the informed trader faces at \( T \) from the final SMO.

Similar to Corollary 6, the next result shows that both the components of the bid-ask spread are equal to certain information decay costs. The slippage component is equal to the information decay cost until the arrival of the final SMO, while the adverse selection is equal to the information decay cost due to the final SMO.

**Corollary 7.** Define the following cost functions:

- **Slippage Cost** \( w^s = (w - g^s(\rho, w)) \sigma_E \)
- **Adverse Selection Cost** \( w^a = -g^a(\rho, w) \sigma_E \)

Then, the two components of the bid-ask spread satisfy

\[
 s^s = \text{Slippage Cost}_\alpha, \quad s^a = \text{Adverse Selection Cost}_\alpha. \tag{22}
\]

**Figure 3: Components of the Bid-Ask Spread and the Slippage Rate.** This figure plots the bid-ask spread \( s \), as well as the slippage component \( s^s \) and the adverse selection component \( s^a \). On the horizontal axis is the informed share \( \rho = 0.05, 0.10, \ldots, 0.95 \). The bid-ask spread and its components are written in units of the price impact parameter \( \Delta \).

The next numerical result shows how the bid-ask spread and its components depend on the informed share \( \rho \).

**Result 2.** The bid-ask spread \( s \), the slippage component \( s^s \), and the adverse selection component \( s^a \) are all positive. Moreover, as functions of the informed share \( \rho \), \( s \) and \( s^s \) are decreasing in \( \rho \), while \( s^a \) is increasing in \( \rho \).
Figure 3 displays the bid-ask spread and its components against the informed share $\rho$. The bid-ask spread and its components are expressed in $\Delta$-units, meaning that we consider the ratios $s/\Delta$, $s^s/\Delta$, and $s^a/\Delta$. Using (5) and Definition 2, we compute

$$\frac{s}{\Delta} = (\alpha - g(\rho, \alpha))\beta \rho^{-1}, \quad \frac{s^s}{\Delta} = (\alpha - g^s(\rho, \alpha))\beta \rho^{-1}, \quad \frac{s^a}{\Delta} = -g^a(\rho, \alpha)\beta \rho^{-1}. \quad (23)$$

The normalization by $\Delta$ does not affect our inferences, because $\Delta$ is independent of $\rho$ (see Proposition 1). We note that all three terms in (23) contain the factor $\beta \rho^{-1} = \frac{\sigma_E}{\Delta}$, which is strongly decreasing in $\rho$. This is because, as discussed in Section 4.1, the market is more informationally efficient when there are more informed traders, which translates into the efficient volatility $\sigma_E$ becoming smaller. One may thus expect that all three terms in (23) are decreasing in $\rho$. Result 2 shows that this is indeed true for the bid-ask spread $s$ and the slippage component $s^s$, but not for the adverse selection component $s^a$.

The adverse selection component of the bid-ask spread, $s^a$, reflects the fact that the initial BLO is eventually executed by a SMO coming potentially from a future informed trader, with superior information. Thus, as expected, adverse selection increases when there are more future informed traders, that is, $s^a$ is increasing in the informed share $\rho$. Moreover, $s^a$ is close to zero when the $\rho$ approaches zero. This is intuitive, since when there are few informed traders, there is little adverse selection.\(^{27}\)

The slippage component of the bid-ask spread, $s^s$, reflects the phenomenon of slippage, which is the signal decay caused by competition with the other informed traders. When there are more informed traders (the informed share $\rho$ is higher), more informed traders are likely to arrive in the future, and therefore the rate at which slippage occurs is higher. The total amount of slippage, however, is multiplied by the efficient volatility $\sigma_E$ (recall that signals are normalized by the efficient volatility). Since the efficient volatility is strongly decreasing in the informed share, the slippage component is actually decreasing in the informed share, as can be seen in Figure 3.

The bid-ask spread $s$ is the sum of the adverse selection component and the slippage component. In Figure 3 we see that $s$ is indeed decreasing in the informed share $\rho$, although the overall effect is not as strong as the effect of $\rho$ on efficient volatility. When

\(^{27}\)In our model, informed traders only observe the asset value once, when they arrive to the market. We could set up the model so that informed traders continuously observe the fundamental value. In that case, the adverse selection cost is zero, since all informed traders have the same information. We conjecture, however, that the slippage cost remains positive, as competition among the informed traders still imposes a cost on the submission of limit orders.
the informed share increases from $\rho = 0.05$ to $\rho = 0.95$, the bid-ask spread decreases by about 25%. When the informed share $\rho$ is small, the adverse selection component is close to zero, and therefore most of the bid-ask spread is determined by the slippage component.

Note that the slippage of limit orders can be interpreted as an *endogenous waiting cost* for the informed trader who decides to submit a limit order. Indeed, even though the actual waiting cost of a patient investor is zero, the informed investors’ expected payoff decreases gradually over time because of slippage.\(^{28}\)

The bid-ask spread can be used to construct a clean empirical proxy for the informed share $\rho$ (which does not depend on other parameters such as the fundamental volatility $\sigma_v$ or trading activity $\lambda$). Using equation (5), we compute the ratio of inter-trade volatility $\sigma = \sigma_v / \sqrt{\lambda}$ to the bid-ask spread $s$,

$$\frac{\sigma}{s} = \frac{\rho \sqrt{1 + \gamma^2}}{\alpha - g(\rho, \alpha)}. \quad (24)$$

By taking this ratio, the dependence of both $\sigma$ and $s$ on the other parameters of the model is removed. The inter-trade volatility $\sigma$ does not depend on $\rho$, while Result 2 implies that $s$ is decreasing in $\rho$. Therefore, the ratio $\sigma/s$ is increasing in $\rho$.

We summarize the effect of the informed share $\rho$ on our first two liquidity measures: (i) the equilibrium price impact $\Delta$ is independent on $\rho$, while (ii) the equilibrium bid-ask spread $s$ is decreasing in $\rho$. The reason for this difference is that the price impact is determined by the uninformed traders, while the bid-ask spread is determined by the informed traders. Indeed, the price impact is determined by the evolution of the efficient price over time, which in turn is determined by the uninformed traders. And, since the equilibrium is pooling (Corollary 4), the exact breakdown between informed and uninformed traders becomes irrelevant. By contrast, the bid-ask spread is determined by the optimal order choice of the informed traders, and their order submission strategy can be shown to be elastic in the bid-ask spread. Thus, the share of informed traders affects the equilibrium bid-ask spread.

\(^{28}\)Note that a larger informed share implies higher endogenous waiting costs for an informed trader, *holding the mispricing volatility constant*. However, the informed trader gradually becomes less informed, and thus her mispricing variance increases over time. Therefore, the exact behavior of the average waiting costs is ambiguous, and we leave this analysis for future research.
4.4 Resiliency

The third dimension of liquidity we consider is market resiliency. Kyle (1985) defines resiliency as “the speed with which prices recover from a random, uninformative shock.” Because, as we will see shortly, in our model the speed of price correction is nonlinear in the size of the shock, we define resiliency as the rate at which a small uninformative shock is corrected, in the limit when the size of the shock approaches zero.

More formally, we define resiliency from the point of view of an econometrician who observes a small uninformative shock to the efficient price. Before the shock, the econometrician has the same belief about the asset value as the uninformed traders (the efficient density). Suppose at date \( t \) the efficient price shifts down by a small positive amount \( x \), while the efficient volatility remains the same \((\sigma_E)\). We do not specify the cause of this price shift, but one can imagine it as the reaction to negative public news. The econometrician, however, knows that the shock \( x \) is uninformative, and expects the mispricing to be corrected. We then define resiliency as the rate at which the mispricing is corrected, in the limit when the shift \( x \) approaches zero.

**Definition 3.** Suppose before trading at \( t \), the econometrician believes the asset value has a normal density \( v_t \sim \mathcal{N}(v_t^E+x,\sigma_E^2) \), or equivalently perceives a mispricing \( v_t - v_t^E \sim \mathcal{N}(x,\sigma_E^2) \), with \( x \in \mathbb{R} \). Denote by \( f(x) \) the average mispricing \( v_{t+1} - v_{t+1}^E \) after observing the order at \( t \). The market resiliency coefficient \( K_0 \) is defined by

\[
K_0 = 1 - f'(0).
\]  

(25)

Intuitively, if the order at \( t \) comes from an informed trader, she is more likely to observe a positive mispricing, just as the econometrician does. Hence, she is more likely to submit a buy order, which pushes up the efficient price and reduces the mispricing. Thus, the econometrician expects the forecast error to become smaller on average, which for a positive shock \( x \) translates into \( 0 < f(x) < x \). If we linearize \( f \) near \( x = 0 \), we get \( f(x) \approx f'(0)x = (1-K_0)x \). Hence, \( K_0 \) is the rate at which the mispricing \( x \) gets corrected when \( x \) is small, which is indeed our definition of resiliency. Note that the mechanism behind resiliency is essentially the same as the mechanism behind slippage: the existence of informed traders corrects mispricings over time (resiliency), which generates a cost for an informed trader who submits a limit order (slippage).
Proposition 4. The market resiliency coefficient equals

\[ K_0 = \frac{2(\gamma \phi(0) + (1 - \gamma)\phi(\alpha))}{\beta} \rho^2 \approx 0.8606 \rho^2. \]  

Proposition 4 implies that the market is more resilient when the informed share is larger. This confirms our intuition that a larger share of informed traders results in a faster correction of pricing errors.

Market resiliency is related to information efficiency. Indeed, the market resiliency coefficient \( K_0 \) from equation (26) is proportional to the information precision measure, \( 1/\sigma_E^2 \). Thus, in our model, a larger share of informed traders \( \rho \) causes the market to be both more resilient and more informationally efficient.

5 Robustness and Extensions

In this section, we study the effect of relaxing several assumptions of the benchmark model described in Section 2. Our principle is to relax only one assumption at a time, in order to study the robustness of the model to each of the assumption, while still keeping the model tractable. In the process, we also obtain additional useful empirical predictions.

5.1 Non-Steady-State Equilibrium

In this section, we consider the same benchmark model as in Section 2, but we study a setup in which the efficient volatility is initially different than the parameter \( \sigma_E \) (the efficient volatility in the steady-state equilibrium of Section 3). In this case, the efficient volatility \( \sigma^E \) is no longer constant. Over time, however, we show that the equilibrium converges to the steady-state equilibrium.

We start by defining parameters similar to those in Section 2.1, except that now they depend not only on the informed share \( \rho \), but also on the efficient volatility \( \sigma^E \), or equivalently on the normalized efficient volatility, \( \tau = \sigma^E/\sigma_E \), which is the efficient volatility \( \sigma^E \) divided by the parameter \( \sigma_E \) from (5). To indicate this dependence, we add a tilde over the parameters, and write \( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{g}, \tilde{\Delta}, \) and \( \tilde{s} \). A further complication is that the information function \( \tilde{g} \) cannot be defined independent of the other parameters, as in Definition 1. Therefore, in Definition 4 we define all parameters at the same time.
Recall that \( \phi(\cdot) \) is the standard normal density, and \( \Phi(\cdot) \) is its cumulative density. In the next definition, we also use the parameters \( \alpha, beta, gamma, \sigma_E, Delta, \) and \( s \) from Section 2.1.

**Definition 4.** Let \( \rho > 0 \) and \( \tau > 0 \). We define the functions \( \tilde{alpha} = \tilde{alpha}(\rho, \tau), \tilde{beta} = \tilde{beta}(\rho, \tau), \tilde{gamma} = \tilde{gamma}(\rho, \tau), \) and \( \tilde{g} = \tilde{g}(\rho, w, \tau) \) by the implicit equations:

\[
\begin{align*}
\tilde{alpha} - \tilde{g}(\rho, \tilde{alpha}, \tau) &= \alpha - g(\rho, \alpha) - 2 \frac{\rho \tau \phi(\frac{\alpha}{\tau})}{1 + \rho(1 - \Phi(\frac{\alpha}{\tau}))}, \\
\tilde{beta} &= \frac{1 - \rho}{\phi(\frac{\alpha}{\tau})} + \frac{1 - \rho}{\phi(\frac{\alpha}{\tau})} \frac{1 - \rho}{\phi(\frac{\alpha}{\tau})} \\
\tilde{gamma} &= \frac{\phi(0) - \phi(\frac{\alpha}{\tau})}{1 + \rho(1 - \Phi(\frac{\alpha}{\tau}))},
\end{align*}
\]

while \( \tilde{g}(\rho, w, \tau) \) is defined as in Definition 1 by adding a tilde over the corresponding parameters,\(^{29}\) and letting \( \tau \) evolve according to \( \tau_0 = \tau \) and

\[
\tau_{t+1}^2 = \rho^2 \frac{1 + \gamma^2}{2\beta^2} + \tau_t^2 - 2\rho^2\tau_t^2 \left( \frac{\phi(\frac{\alpha}{\tau})}{1 + \rho(1 - \Phi(\frac{\alpha}{\tau}))} + \frac{\phi(\frac{\alpha}{\tau}) - \phi(\frac{\alpha}{\tau})}{1 + \rho(1 - \Phi(\frac{\alpha}{\tau}))} \right),
\]

where \( \alpha_t = \tilde{alpha}(\rho, \tau_t) \). Also, define \( \sigma_E^\prime, \Delta_t, \tilde{g}, \tilde{s} \) by

\[
\sigma_E^\prime = \tau \sigma_E, \quad \Delta_t = \frac{\rho}{\beta} \sigma_E^\prime, \quad \tilde{s} = (\tilde{alpha} - \tilde{g}(\rho, \tilde{alpha}, \tau)) \sigma_E.
\]

When \( \tau = 1 \), one can check that the values of \( \tilde{alpha}, \tilde{beta}, \tilde{gamma}, \tilde{g}, \Delta, \) and \( \tilde{s} \) are equal to the corresponding values without the tilde, which are related to the steady-state equilibrium of Section 3. Moreover, when \( t \) is large, \( \tau_t \) approaches 1, indicating that the equilibrium of this section becomes steady-state over time. This is confirmed by Result 3 below.

Proposition 5 shows that there exists a Markov Perfect Equilibrium (MPE) of the model if the conditions in Result IA.3 in the Internet Appendix are satisfied. We verify these conditions numerically in Internet Appendix Section 4. The next result also describes several properties of the equilibrium.

**Proposition 5.** If the conditions in Result IA.3 are satisfied, there exists a Markov Perfect Equilibrium of the game. Suppose at \( t = 0 \) the efficient price is \( v_0^E \), and the normalized efficient volatility is \( \tau_0 > 0 \). Then, in equilibrium the normalized efficient volatility \( \tau_t = \sigma_t^E / \sigma_E \) evolves according to the equation (28). Let \( \Delta_t, \gamma_t, \) and \( s_t \) be as in Definition 4. Then, an order arriving at \( t \) changes the efficient price from \( v_t^E \) to...

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\(^{29}\)The only case when a tilde is not added over a parameter is when the term \( \rho \sqrt{\frac{1 + \gamma^2}{2\beta^2}} \) occurs. This term is equal to \( \frac{s}{\sigma_E} \), the normalized inter-trade variance, where \( \sigma \) is the volatility of the change in fundamental value between trades.
In equilibrium, the bid-ask spread and the price impact of an order are no longer constant. The next result, however, provides a linear combination that remains constant over time. The result involves the parameters $s$ and $\Delta$ from equation (5).

**Corollary 8.** The equilibrium bid-ask spread $\tilde{s}_t$ and price impact coefficient $\tilde{\Delta}_t$ satisfy

$$
\tilde{s}_t - \tilde{\Delta}_t = \frac{s}{2} - \Delta.
$$

Equation (30) is the indifference condition for the uninformed traders. Consider an uninformed trader who is the first in the bid queue, and suppose his BLO is executed at date $t$ by a SMO. Then, net of his private valuation, his expected payoff is $\tilde{s}_t/2 - \tilde{\Delta}_t$, where $\tilde{s}_t/2$ represents the difference between the efficient price and the bid price, and $\Delta$ represents the adverse selection loss from a potentially informed SMO. If his expected payoff were not the same at $t + 1$, then the uninformed traders would have an incentive to modify their position in the bid queue. The discussion thus far explains why the expected payoff $\tilde{s}_t/2 - \tilde{\Delta}_t$ is constant. The fact that it is equal to $\tilde{s}/2 - \tilde{\Delta}$ reflects the fact that the equilibrium converges to the steady-state equilibrium of Section 3. We state this as a numerical result.

**Result 3.** As $t$ becomes large, the efficient volatility $\sigma^E_t$ approaches the parameter $\sigma_E$, the bid-ask spread $\tilde{s}_t$ approaches $s$, and the price impact coefficient $\tilde{\Delta}_t$ approaches $\Delta$.

We now analyze the speed of convergence to the steady-state equilibrium, and describe the equilibrium in more detail, in particular regarding the market quality measures introduced in Section 4: information efficiency, price impact, bid-ask spread, market resiliency. In addition, we introduce a new measure, the market-to-limit ratio, which is defined as the probability the next order is a market order divided by the probability that the next order is a limit order.

As in Section 4, we begin with an analysis of the efficient volatility, $\sigma^E_t$. Recall that in the steady-state equilibrium of Section 3, the efficient volatility is constant, and equal to the parameter $\sigma_E$. In the non-steady-state case, the efficient volatility is no longer constant, but Result 3 shows that it converges to the steady-state value. Therefore, if
**Figure 4: Dynamic Information Efficiency.** This figure plots the evolution of the normalized efficient volatility over time in the non-steady-state equilibrium. The normalized efficient volatility ($\tau$) is the efficient volatility ($\sigma^E$) divided by its steady-state value ($\sigma_E$). On the horizontal axis is time in logarithmic scale. Each curve in the plot corresponds to a value of the informed share ($\rho$) ranging from 0.05 to 0.95. For each line, the starting normalized efficient volatility is $\tau = 2$.

we define the normalized efficient volatility as the ratio $\tau_t = \sigma^E_t / \sigma_E$, it should converge to one. Figure 4 displays the evolution over time of the normalized efficient volatility, according to equation (28). Each curve in the plot corresponds to a value of the informed share $\rho$ ranging from 0.05 to 0.95. We observe that in all cases the normalized efficient volatility indeed converges to one, and furthermore, that the speed of convergence is inversely related to the informed share.

More formally, we define the *recovery time* to be the number of trading rounds it takes for the normalized efficient volatility to revert within a neighborhood of one after a positive or negative shock. In Figure 4, we choose as neighborhood of one the interval $(1 - 10^{-4}, 1 + 10^{-4})$, while the initial value after the shock is $\tau = 2$. Numerically, the recovery time appears to be linear in the inverse informed share, $1/\rho^2$, regardless of the choice of neighborhood or shock size. We report this fact as a numerical result.

**Result 4.** The recovery time after a shock to the normalized efficient volatility ($\sigma^E_t / \sigma_E$) is linear in the inverse squared informed share ($1/\rho^2$).
This result confirms our previous intuition that informed traders make the market more dynamically efficient. Indeed, when there are more informed traders (the informed share is higher), a shock that moves the efficient volatility away from its steady-state value ($\tau = 1$) is followed by a quicker reversal to the steady-state value, and hence it requires a shorter recovery time. The quicker convergence occurs because orders carry more information when the informed share is higher, since the probability of each order being submitted by an informed trader is higher.

The inverse recovery time is thus a measure of information efficiency, and Result 4 shows that this measure is linear in the squared informed share ($\rho^2$). In Section 4.1, another measure of information efficiency is the inverse steady-state efficient variance ($1/\sigma_E^2$), which is also linear in the squared informed share.\textsuperscript{30} The two measures share the same dynamic efficiency intuition, but the inverse recovery time measure is more explicit in how dynamic efficiency is achieved.

**Figure 5: Market Quality in the Non-Steady-State Equilibrium.** This figure plots two market quality measures in the non-steady-state equilibrium. On the horizontal axis is the normalized efficient volatility ($\tau$), which is the efficient volatility ($\sigma^E$) divided by its steady-state value ($\sigma_E$). Each curve in the plots corresponds to a value of the informed share ($\rho$) ranging from 0.05 to 0.95. The left graph plots the relative price impact coefficient, which is the price impact coefficient ($\Delta$) divided by its steady-state value ($\Delta$). The right graph plots the market-to-limit ratio ($P_{MO}/P_{LO}$), which is the probability the next order is a market order, divided by the probability that the next order is a limit order. If the numerical procedure does not yield a unique value, the corresponding point in the plot is omitted.

\textsuperscript{30}By equation (12), the inverse steady-state efficient variance is proportional to $\rho^2$. 

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We now discuss the price impact coefficient, $\hat{\Delta}$, which is the price impact of a buy market order. We define the relative price impact coefficient to be the ratio of $\hat{\Delta}$ by its steady-state value, the parameter $\Delta$. Equation (29) then implies that the relative price impact coefficient is

$$
\frac{\hat{\Delta}}{\Delta} = \frac{\tilde{\beta}(\rho, \tau)}{\beta} \tau,
$$

(31)

which is a function of the informed share $\rho$ and the normalized efficient volatility $\tau$. The left graph in Figure 5 plots the dependence of the relative price impact coefficient on both $\rho$ and $\tau$. Each curve in the plot corresponds to a value of the informed share $\rho$ ranging from 0.05 to 0.95. We observe that in all cases the relative price impact is increasing in $\tau$. We report this fact as a numerical result. Since $\Delta$ does not depend on either $\rho$ or $\tau$, the next result is equally true for the price impact coefficient $\tilde{\Delta}$.

**Result 5.** The price impact coefficient ($\hat{\Delta}$) and the bid-ask spread ($\tilde{s}$) are increasing in the normalized efficient volatility ($\tau$).

Intuitively, when the normalized efficient volatility $\tau$ is larger, the uninformed traders have imprecise knowledge about the fundamental value, and therefore the adverse selection is stronger. That implies that the price impact of a buy market order, $\tilde{\Delta}$, is larger, as confirmed by Result 5. The bid-ask spread, $\tilde{s}$, is also larger, to compensate the uninformed traders for the increase in adverse selection. Formally, the bid-ask spread and price impact vary with $\tau$ in the same way, since equation (30) implies that the difference $\tilde{s}/2 - \tilde{\Delta}$ is equal to $s/2 - \Delta$, which does not depend on $\tau$.

Putting together the previous results, it follows that after a positive shock in the efficient volatility (or equivalently in $\tau$), the bid-ask spread $\tilde{s}$ initially increases, to adjust for the higher value of $\tau$, after which it decreases gradually to its steady-state value, $s$. The same effect occurs for the price impact $\hat{\Delta}$. Thus, in our model the bid-ask spread and the price impact coefficient both display resiliency, in the sense that they eventually recover to their steady-state values after a shock in the efficient volatility. We call this phenomenon liquidity resiliency.

Liquidity resiliency is different from market resiliency. As discussed in Section 4.4, market resiliency is defined as the recovery of prices after an uninformative shock. In

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31The figure displays results computed with the function $g$ instead of $\tilde{g}$; very similar results are obtained by using instead the estimated function $\hat{g}$. The numerical procedure used to solve for the equilibrium is explained in Internet Appendix Section 4. We impose the strict condition that the solution to the first equation in (27) must be unique. When the threshold $\bar{a}$ is close to zero, this condition is not satisfied because of estimation errors. This explains why there are missing points in Figure 5.
the context of the non-steady-state equilibrium, we obtain the following result similar to Proposition 4.

**Proposition 6.** The market resiliency coefficient in the non-steady-state equilibrium equals

\[ \tilde{K}_0 = \frac{2 \rho^2}{\tau} \frac{\bar{\gamma} \phi(0) + (1 - \bar{\gamma}) \phi\left(\frac{\bar{\gamma}}{\tau}\right)}{\beta}. \]  

(32)

Numerically, the market resiliency coefficient \( K_0 \) is increasing in the informed share \( \rho \), and decreasing in the normalized efficient volatility \( \tau \). The intuition for why market resiliency is increasing in the informed share is the same as in the steady-state equilibrium. The new result is that market resiliency is decreasing in the normalized efficient volatility. Intuitively, when the efficient volatility is large, the informed traders become less aggressive and are more likely to submit limit orders (as we explain below). Therefore, it takes longer for the price to converge to the fundamental value.

We introduce a new market of market quality, the market-to-limit ratio, which is defined as the probability the next order is a market order, divided by the probability that the next order is a limit order. In the steady-state equilibrium of Section 3, this ratio is equal to one since all types of orders are equally likely (see Corollary 3). In the non-steady-state equilibrium, the market-to-limit ratio varies with both the informed share and the normalized efficient volatility.

**Proposition 7.** The market-to-limit ratio in the non-steady-state equilibrium equals

\[ \frac{P_{MO}}{P_{LO}} = \frac{\frac{1 - \rho}{4} + \rho (1 - \Phi(\frac{\bar{\gamma}}{\tau}))}{\frac{1 - \rho}{4} + \rho (\Phi(\frac{\bar{\gamma}}{\tau}) - \Phi(0))}. \]  

(33)

The right graph in Figure 5 plots the dependence of the market-to-limit ratio on both \( \rho \) and \( \tau \). Each curve in the plot corresponds to a value of the informed share \( \rho \) ranging from 0.05 to 0.95. We observe that in all cases the market-to-limit ratio is decreasing in \( \tau \). Intuitively, as explained before, when the normalized efficient volatility \( \tau \) is larger, there is an increase in adverse selection for the uninformed traders. This causes the bid-ask spread, as well as the price impact, to be larger. But the increase in the bid-ask spread changes the informed traders’ tradeoff between market orders and limit orders, and makes limit orders more attractive. Thus, the market-to-limit ratio is smaller when the efficient volatility is larger. This result, along with our previous results regarding the resiliency of the bid-ask spread and the price impact after a efficient volatility shock
provide new testable empirical implications.

5.2 Public Information Processing

In the benchmark model, the uninformed traders can compute the first moment of the posterior belief of the asset value conditional on the order flow, and the unconditional second moment of the posterior belief, but they cannot compute any higher moments. In this section, we assume that the uninformed traders can compute the entire posterior belief.

As in the previous sections, we define the efficient density to be the uninformed traders’ belief about the fundamental value just before trading at $t$. Let $v_t^E$ and $\sigma_t^E$ be respectively the mean and the volatility of the efficient density. Because the uninformed traders can compute all moments of the distribution, the shape of the efficient density is no longer constant and normal as in the benchmark model, but it keeps changing after each order. Nevertheless, at the initial date $t = 0$ the efficient density is assumed to be $\mathcal{N}(0, \sigma_E^2)$, which is normal with standard deviation equal to the efficient volatility parameter $\sigma_E$ from (5). Equivalently, if we define the normalized efficient density $\psi_t$ as the density of the signal $w_t = \frac{v_t - v_t^E}{\sigma_E}$, then the initial normalized efficient density is standard normal: $\psi_0 = \mathcal{N}(0, 1)$.

The other assumptions are as in the benchmark model. In particular, the informed trader can compute only the average signal at the time of her limit order’s execution. In the benchmark model, this expectation is equal to the information function $g$ from Definition 1. We assume that in the current setup, the informed traders use the same information function when computing their expected payoff from a limit order.\textsuperscript{32}

Proposition 8 shows that there exists a Markov Perfect Equilibrium (MPE) of the model if the conditions in Result IA.4 Internet Appendix are satisfied. These conditions are verified in Internet Appendix 4. The next result also describes several properties of the equilibrium.

**Proposition 8.** If the conditions in Result IA.4 are satisfied, there exists a Markov Perfect Equilibrium of the game. In equilibrium, if an order $O \in \{\mathrm{BMO}, \mathrm{BLO}, \mathrm{SLO}, \mathrm{SMO}\}$ arrives at $t$, the efficient price changes from $v_t^E$ to $v_t^E + \Delta \psi_t, O$, and the efficient density

\textsuperscript{32}This is consistent with the principle that information processing is difficult. The computation of the exact information function in the current setup would be much more difficult, because the information function now depends on an additional parameter: the efficient density $\psi$, which keeps changing shape with each order.
changes from $\psi_t$ to $\psi_{t+1} = f_{\psi_t, O}$, where $\delta_{\psi, O} = \frac{\Delta_{\psi, O}}{\sigma_E}$ and $f_{\psi, O}$ are defined in equation (IA.54) in the Appendix. At date $t$, the ask price is $v_t^E + H_t^a$ and the bid price is $v_t^E - H_t^b$, where $H_t^a = s/2 + \Delta_{\psi_t, BMO} - \Delta$ and $H_t^b = s/2 - \Delta_{\psi_t, SMO} - \Delta$.

We call $H_t^a$ the ask half-spread and $H_t^b$ the bid half-spread. These quantities are not equal to each other, because in the current context the efficient density is now precisely computed by the uninformed traders, and is therefore no longer normal. The exact shape of the posterior density and its evolution are given in Proposition 8.

The last statement in Proposition 8 is the indifference condition for the uninformed traders. Consider an uninformed trader who is first in the ask queue at date $t$. If a buy market order arrives, then his expected payoff (net of his private valuation) is given by the ask half-spread ($H_t^a$), minus the adverse selection of the buy market order ($\Delta_{\psi_t, BMO}$). The expected payoff is the same regardless of the date $t$, because otherwise the uninformed traders would have an incentive to modify their position in the bid queue. The discussion thus far explains why the expected payoff $H_t^a/2 - \Delta_{\psi_t, BMO}$ is constant. We set this constant equal to $s/2 - \Delta$ because we want the equilibrium to be on average the same as the steady-state equilibrium of Section 3. More directly, this equality holds if the initial normalized efficient density $\psi_0$ is set to be the standard normal density $N(0, 1)$.

In equilibrium the informed trader has the same threshold strategy as in Corollary 2, because she uses the same information function $g$ as in the steady-state equilibrium. Depending on her signal $w_t = \frac{v_t - v_t^E}{\sigma_E}$, she submits a BMO if $w_t \in (\alpha, \infty)$, BLO if $w_t \in (0, \alpha)$, SLO if $w_t \in (-\alpha, 0)$, or SMO if $w_t \in (-\infty, -\alpha)$. The magnitude of the price impact thus depends on how the normalized efficient density $\psi_t$ is averaged out over the intervals that define the informed trader’s strategy.

Figure 6 plots the relative price impact coefficient after observing a particular order flow sequence of length 1 and 2. The relative price impact of a buy market order is the ratio $\Delta_{\psi_t, BMO}/\Delta$, where $\Delta$ is the price impact of a BMO in the benchmark model, and $\psi_t$ is the efficient density after the particular order flow sequence, assuming that the initial density is standard normal. The left graph in Figure 6 plots the relative price impact for the four types of order after a BMO is observed (the four points on the left) or after a BLO is observed (the four points on the right). There is no need to analyze the relative price impact for SLO and SMO, because the results are symmetric with respect

\[33\] This is the equivalent of Corollary 8 in the current context.
**Figure 6: Price Impact with Exact Densities.** This figure plots the relative price impact of an order \( O \in \{ \text{BMO, BLO, SLO, SMO} \} \), conditional on a sequence of orders being observed. The relative price impact of an order \( O \) is the price impact coefficient of that order, divided by the price impact of the same type of order in the benchmark model. For all graphs, the initial normalized efficient density (before the sequence of orders) is the standard normal density. The left graph plots the relative price impact coefficient of the order \( O \) after an order \( O_1 \) is observed, where \( O_1 \) is either BMO or BLO. The right graph plots the relative price impact coefficient of the order \( O \) after two orders \( O_1 \) and \( O_2 \) are observed, where \( O_1 \in \{ \text{BMO, BLO} \} \) and \( O_2 \in \{ \text{BMO, BLO, SLO, SMO} \} \). The informed share is \( \rho = 0.10 \).

To the line \( y = 1 \). We observe that in all cases the price impact coefficients are close to the benchmark price impact, indicating that the standard normal approximation is good.

The right graph in Figure 6 plots the relative price impact for the four types of order after observing a sequence of two orders, \( O_1 \in \{ \text{BMO, BLO} \} \) and \( O_2 = \{ \text{BMO, BLO, SLO, SMO} \} \). The figure suggests testable implications of the model. To give an example, after an order flow sequence containing a BLO, the price impact of a BMO is generally lower than in the benchmark model. For instance, after observing two BLOs in a row, the relative price impact of a BMO is significantly smaller than one. The intuition is that the occurrence of a BLO indicates to the uninformed traders that the asset mispricing is relatively smaller (that is, its standard deviation is lower) than after observing a market order. Therefore, a subsequent BMO is less likely to be informed and hence has a smaller price impact.

Overall, we conclude that the price impact of various types of orders stays close to the price impact in the benchmark model. To provide further evidence, consider Figure 7.
Figure 7: Average Efficient Density. Suppose the density of the signal $w_t = \frac{v_t - v^E_t}{\sigma_E}$, also called the (normalized) efficient density, is standard normal. Then, this figure plots the efficient density at $t + 1$ (i) after observing a BMO (upper left graph), (ii) after observing a BLO (upper right graph), (ii) the average density after observing a BMO or a BLO (lower left graph), and (iv) the average density after observing a BMO, BLO, SLO, or a SMO (lower right graph). In all graphs, the efficient density is plotted with a solid line, while the standard normal density is plotted with a dashed line. The informed share is $\rho = 0.10$.

The upper graphs plot the normalized efficient density after a BMO and after a BLO.\textsuperscript{34} We see in both cases that the shape of the posterior density is not normal, although the difference from the standard normal density is not large. If we take the average of these shapes (the two lower graphs in Figure 7), we see that the average efficient density is quite close to the benchmark efficient density, which is standard normal. We obtain very similar results if instead of taking the average efficient density after one order, we take the average after several orders. All these results suggest that our assumption that the uninformed traders process information by using a normal approximation to the efficient density is plausible.

\textsuperscript{34}The normalized efficient densities after SMO and SLO are, respectively, symmetric around the $y$-axis to the densities for BMO and BLO.
5.3 No-Trade Region

In the benchmark model, each informed trader receives a small penalty $\nu$ (called the “commitment parameter”) if after observing the fundamental value she chooses not to trade. In this section, we set $\nu = 0$. Because in this case the informed trader might choose not to submit any order, we assume that whenever this event occurs, another trader is drawn instantly from the pool. Since informed traders are identical, we can directly assume that whenever an informed trader decides not to trade, she is immediately replaced by an uninformed trader.\footnote{This assumption is consistent with the principle of working in event time rather than calendar time. Indeed, if an informed trader decides not to trade at $t$, the clock only gets restated when an uninformed trader arrives to the market and trades, in which case this event occurs at $t+1$. If we worked in calendar time instead, the model would be more complicated, because the time elapsed between trades would become informative about the fundamental value.}

In this section we use a methodology as in the benchmark model, except that we must account for the fact that the optimal strategy of the informed trader includes a “no-trade region” when the informed share $\rho$ is above a threshold. Thus, instead of the parameter $\alpha$ (the threshold signal between BLO and BMO), we introduce two parameters that are functions of $\rho$. The first parameter, $\alpha^0$, is the threshold signal between not trading and BLO, with $\alpha^0 = 0$ if the no-trade region is empty. The second parameter, $\alpha_1$ is the threshold signal between BLO and BMO, with $\alpha_1 = \alpha$ if the no-trade region is empty.

We start by defining parameters similar to those in Section 2.1, except that all parameters now depend on the informed share $\rho$. To indicate the different values, we add a superscript “1” to the parameters, and write $\alpha^1$, $\beta^1$, $\gamma^1$, $g^1$, $\Delta^1$, $\sigma_E^1$, and $s^1$. Also, the new information function $g^1$ cannot be defined independent of the other parameters, as in Definition 1. Therefore, in Definition 5 we define all parameters at the same time. Recall that $\phi(\cdot)$ is the standard normal density, and $\Phi(\cdot)$ is its cumulative density. In the next definition, we also use the parameters $\alpha$, $\beta$, $\gamma$, $g$, $\sigma_E$, $\Delta$, and $s$ from Section 2.1.

**Definition 5.** Let $\rho > 0$. We define the functions $\alpha^1 = \alpha^1(\rho)$, $\beta^1 = \beta^1(\rho)$, $\gamma^1 = \gamma^1(\rho)$, and $g^1 = g^1(\rho, w)$ by the implicit equations:

$$
\alpha^1 - g^1(\rho, \alpha^1) + 2g^1(\rho, \alpha^0) = 0, \quad 1 - \Phi(\alpha^1) = \Phi(\alpha^1) - \Phi(\alpha^0),
$$

$$
\beta^1 = \frac{1}{4\phi(\alpha^1)}, \quad \gamma^1 = \frac{\phi(\alpha^0) - \phi(\alpha^1)}{\phi(\alpha^1)},
$$

where $g^1(\rho, w)$ is as in Definition 5 by adding a superscript “1” to the corresponding
parameters, and replacing equation (3) with

\[
\pi_{\psi, O} = \int \frac{1 - \rho}{4} + \frac{\rho}{4} \mathbf{1}_{z \in i_O} + \rho \mathbf{1}_{z \in i_O} \right) \psi(z) \, dz,
\]

\[
f_{\psi, O}(x) = \int \frac{1 - \rho}{4} + \frac{\rho}{4} \mathbf{1}_{z \in i_O} + \rho \mathbf{1}_{z \in i_O} \right) \psi(z) \phi \left( x; z - \delta, \rho \sqrt{\frac{1 + (\gamma^2)}{2(\beta)}} \right) \, dz,
\]

where \( i_O = \{(\alpha^1, \infty), (\alpha^{0}, \alpha^{0}), (-\alpha^0, \alpha^0), (-\alpha^1, -\alpha^0), (-\infty, -\alpha^1)\} \) respectively for \( O \in \{BMO, BLO, NO, SLO, SMO\} \). If there is no solution to (34) with \( \alpha^0 > 0 \), define \( \alpha^0 = 0 \), \( \alpha^1 = \alpha \), \( \beta^1 = \beta \), \( \gamma^1 = \gamma \), and \( g^1 = g \).

Also, define \( \Delta^1 \), \( \sigma^1_E \), and \( s^1 \) as follows:

\[
\Delta^1 = \sqrt{\frac{2}{1 + (\gamma^2)}}, \quad \sigma^1_E = \beta^1 \rho^{-1} \Delta^1, \quad s^1 = (\alpha^1 - g^1(\rho, \alpha^1)) \sigma^1_E. \tag{36}
\]

Note that, as in Corollary 3, the condition \( 1 - \Phi(\alpha^1) = \Phi(\alpha^1) - \Phi(\alpha^0) \) from Definition 5 ensures that all orders are equally likely, and that the equilibrium is in steady-state.

Proposition 9 shows that there exists a Markov Perfect Equilibrium (MPE) of the model if the conditions in Result IA.5 in the Internet Appendix are satisfied. We are not able to prove these conditions analytically, but we verify them numerically in Internet Appendix Section 4. The next result also describes several properties of the equilibrium.

**Proposition 9.** If the conditions in Result IA.5 are satisfied, there exists a Markov Perfect Equilibrium of the game. In equilibrium, if an order \( O \in \{BMO, BLO, SLO, SMO\} \) arrives at \( t \), the efficient price changes from \( v^E_t \) to \( v^E_t + \Delta^1_O \), where, respectively, \( \Delta^1_O = \{\Delta^1, \gamma^1 \Delta^1, -\gamma^1 \Delta^1, -\Delta^1\} \). An informed trader who arrives at \( t \geq 0 \) and observes \( w_t = \frac{v^E_t - v^E_t}{\sigma^1_E} \) in the interval \( i_O = \{(\alpha^1, \infty), (\alpha^0, \alpha^0), (-\alpha^0, \alpha^0), (-\alpha^1, -\alpha^0), (-\infty, -\alpha^1)\} \) optimally submits an order \( O \in \{BMO, BLO, NO, SLO, SMO\} \), respectively. At date \( t \), the ask price is \( v^E_t + s^1/2 \) and the bid price is \( v^E_t - s^1/2 \).

In equilibrium, the size of the no-trade region \( (-\alpha^0, \alpha^0) \) depends on the informed share \( \rho \). Result 6 below shows that there exists a threshold informed share \( (\rho_0 \approx 0.1560) \), such that when the informed share is below this threshold the no-trade region is empty. We call this region the low-information regime, as opposed to the case when \( \rho > \rho_0 \) which is called the high-information regime. Result 6 also shows that the size of the no-trade region is increasing in the informed share in the high-information regime. The
**Figure 8: Equilibrium with No-Trade Region.** This figure plots several variables in the equilibrium with no-trade region, against the informed share $\rho$: (i) the threshold signal $\alpha^1$ between BLO and BMO, and the threshold $\alpha^0$ of the no-trade region (between BLO and NO = no order), (ii) the price impact coefficient $\Delta^1$, divided by the benchmark value $\Delta$, (iii) the efficient volatility $\sigma^1_E$, divided by the benchmark value $\sigma_E$, and (iv) the bid-ask spread $s^1$, divided by the benchmark price impact coefficient $\Delta$.

Intuition is that competition among informed traders makes it less profitable to trade when the informed trader’s signal is moderate. Differently said, competition among informed traders raises the bar for their incentive to trade on information.

**Result 6.** There exists a threshold informed share $\rho_0 \approx 0.1560$, such that when $\rho < \rho_0$ (the “low-information regime”) the no-trade region is empty and the equilibrium coincides with the benchmark equilibrium. When $\rho < \rho_0$ (the “high-information regime”), (i) the no-trade region $(-\alpha^0, \alpha^0)$ is nonempty and increasing in $\rho$, (ii) $\alpha^1, \beta^1$ and $\gamma^1$ are increasing in $\rho$, (iii) the relative efficient volatility $\sigma^1_E/\sigma_E$ is increasing in $\rho$, (iv) the price impact coefficient $\Delta^1$ is decreasing in $\rho$.
Figure 8 illustrates graphically some stylized facts from Result 6. In the high-information regime, adverse selection as measured by the price impact coefficient $\Delta^1$ is decreasing in $\rho$. This is because when competition among informed traders increases, the threshold ($\alpha^0$) for trading on information increases, and as because of the decrease in informed trading activity, the overall adverse selection decreases, although not strongly (by at most 7%). (Recall that the overall adverse selection parameter $\Delta$ is constant in the benchmark model.) The effect on the bid-ask spread is ambiguous. In the benchmark model, the increase in competition among the informed traders makes the market more dynamically efficient and therefore lowers the bid-ask spread (although not very strongly). In the high-information regime, the existence of a no-trade region decreases dynamic efficiency relative to the benchmark model. Indeed, according to Result 6, the relative efficient volatility $\sigma^1_E/\sigma_E$ is increasing with $\rho$. Nevertheless, the efficient volatility itself $\sigma^1_E$ is still decreasing in $\rho$. Overall, the effect of $\rho$ in the high-information regime is very weak, and the bid-ask spread appears approximately constant in this region, although the result is less conclusive because the estimation error is relatively large in this case.

6 Conclusion

We have presented a model of an order driven market with asymmetric information in which investors can choose between demanding liquidity with a market order and providing liquidity with a limit order. Despite the difficulty of the problem, the model is tractable, and—except for the information function that is computed numerically—our results are obtained in closed form.

Our main result is that informed trading, as proxied in our model by the informed share, has an overall positive effect on liquidity, under its three dimensions: tightness (bid-ask spread), depth (price impact), and resiliency (the speed at which pricing errors are corrected). In particular, a larger informed share (i) leads to a smaller bid-ask spread, (ii) generates a stronger market resiliency, yet (iii) does not affect the price impact of one additional unit of trading. From the perspective of the informed trader, limit orders have a slippage cost, which measures the erosion in information advantage due to the competition from future informed traders. Slippage costs represent an endogenous waiting cost for informed traders, and generate a new component of the bid-ask spread.
We also estimate the information content of order flow. In particular, because in equilibrium informed traders also use limit orders—whereas in much of the theoretical literature informed traders only use market orders,—in our model limit orders also have a nonzero price impact. Quantitatively, we find that the price impact of a limit order is roughly one fourth of the price impact of a market order.

The results described thus far assume that the equilibrium is in steady-state. If an uncertainty shock moves the limit order book out of steady-state by suddenly increasing the efficient volatility, then the model predicts that the equilibrium will revert back to steady-state, at a speed that is increasing in the informed share. We introduce a new measure, the market-to-limit ratio, which measures the probability of a trader to submit a market order relative to a limit order. After an uncertainty shock, the market-to-limit ratio drops significantly below one, as the increase in the bid-ask spread (and price impact) temporarily convinces the informed traders to submit more limit orders. The connections among the market-to-limit ratio with the liquidity measures and the efficient volatility, as well as the expected evolution of the equilibrium towards the steady-state, produce new testable implications of the model.

Our results show that informed trading has an important effect on liquidity, especially under its resiliency aspect. But estimating market resiliency directly is difficult, since that would involve having access to information that is not public. Instead, our results regarding out-of-steady-state equilibria suggest that we can use liquidity resiliency instead, which is observable as long as we can identify uncertainty shocks.

Yet another approach is to use rigidities such as stale prices as evidence of low market resiliency, and study the connection with informed trading. We argue that market resiliency is inversely related to the price delay measure of Hou and Moskowitz (2005, in short HM05). HM05 find empirically that firms in which the price responds with a delay to information command a large return premium. Interestingly, HM05 find that the delay premium has little relation with the PIN measure of Easley, Hvidkjaer, and O’Hara (2002), which is another measure of informed trading. This suggests that the informed share in our model may in fact be measuring a different aspect of informed trading than PIN. Since PIN is based on large imbalances between buyers and sellers,

\[36\text{Indeed, it is plausible that firms in which prices respond with a delay to information are also firms for which prices move more slowly toward the fundamental value. It is true that HM05 consider delay at weekly (or in some robustness checks at daily) frequency, while in our model it is more natural to think of events as occurring at higher, intra-day frequencies. Then, our identification is correct if delay at lower frequencies is correlated with delay at higher frequencies.}\]
we postulate that PIN is related to informed trading done by large traders, possibly corporate insiders. By contrast, the informed share in our model may be more related to trading done by small informed traders that are not necessarily insiders, and are just better informed than the public.

Overall, our theoretical model produces a rich set of implications regarding the connection between the activity of informed traders and the level of liquidity. We find that informed traders have on aggregate a positive effect, by making the market more efficient and, at the same time, more liquid. A welfare analysis also shows that a larger number of informed traders (caused for instance by an exogenous decrease in information costs) increases aggregate trader welfare. Our model thus provides useful tools to analyze informed trading, and its connection with liquidity, prices, and welfare.

Appendix. Proofs

Before proving Theorem 1 and discussing Result 1, we explain how investors’ beliefs are updated after observing the order flow. For an order \( O = \{\text{BMO, BLO, SLO, SMO}\} \), define, respectively, \( \delta_O = \left\{ \frac{\gamma}{\beta}, \frac{\gamma}{\beta}, -\frac{\gamma}{\beta}, -\frac{\gamma}{\beta} \right\} \) and \( i_O = \{(-\infty, \alpha), (0, \alpha), (-\alpha, 0), (-\infty, -\alpha)\} \).

Let \( \phi(\cdot; M, S) \) be the normal density with mean \( M \) and standard deviation \( S \), \( \phi(\cdot) \) the standard normal density (\( M = 0, S = 1 \)), and \( \Phi(\cdot) \) the cumulative normal density. Denote the normalized inter-trade volatility by \( \tilde{\sigma} = \frac{\sigma}{\sigma_E} = \rho \sqrt{1 + \frac{\gamma^2}{2\beta^2}}. \)

Lemma A1. In the context of Theorem 1, consider a trader who just before trading at \( t \) believes the signal \( w_t = \frac{v_t - v_t^E}{\sigma_E} = z \) has probability density function \( \psi_t(z) \). Then, the following are true:

(a) The probability of observing \( O \) at \( t \) is

\[
P_O = \frac{1 - \rho}{4} + \rho \int_{z \in i_O} \psi_t(z)dz. \quad (A1)
\]

After seeing the order \( O \) at \( t \), the posterior density of \( w_{t+1} = \frac{v_{t+1} - v_{t+1}^E}{\sigma_E} \) is

\[
\psi_{t+1, O}(x) = \frac{\int \left( \frac{1 - \rho}{4} + \rho 1_{z \in i_O} \right) \psi_t(z) \phi(x; z - \delta_O, \tilde{\sigma})dz}{P_O}. \quad (A2)
\]

(b) Suppose \( \psi_t(\cdot) \) is not necessarily normal, and has mean \( \mu_t \) and standard deviation \( \sigma_t \). Define the normalized price impact \( \delta_{t+1, O} \) to be the change in the expectation
of \( w_t \) after observing \( \mathcal{O} \) at \( t \). Then,

\[
\delta_{t+1, \mathcal{O}} = \mathbb{E}(w_t \mid \psi_t, \mathcal{O}) - \mathbb{E}(w_t \mid \psi_t) = \frac{\rho \int_{\mathcal{O}} \psi_t(z)(z - \mu_t)dz}{P_{\mathcal{O}}}. 
\] (A3)

Denote by \( \mu_{t+1, \mathcal{O}} \) and \( \sigma_{t+1, \mathcal{O}} \) the mean and standard deviation, respectively, of the posterior density \( \psi_{t+1, \mathcal{O}}(x) \). Let \( V_{t+1, \mathcal{O}} = \frac{1}{P_{\mathcal{O}}} \int \psi_t(z) \left( \frac{1-\rho}{4} + \rho \mathbf{1}_{z \in \mathcal{O}} \right) \left( \frac{z - \mu_t}{\sigma_t} \right)^2 - 1 \right) \) \( dz \).

Then,

\[
\mu_{t+1, \mathcal{O}} = \mu - \delta_{t+1, \mathcal{O}}, \quad \sigma_{t+1, \mathcal{O}}^2 = \sigma_t^2(1 + V_{t+1, \mathcal{O}}) + \sigma^2 - \delta_{t+1, \mathcal{O}}^2, 
\] (A4)

\[
\mathbb{E}(w_t \mid \psi_t, \mathcal{O}) = \mu_{t+1, \mathcal{O}} + \delta_{t+1, \mathcal{O}}. 
\] (A5)

Let \( \bar{\mu}_{t+1} = \mathbb{E}_\mathcal{O}(\mu_{t+1, \mathcal{O}}) \) and \( \bar{\sigma}_{t+1}^2 = \mathbb{E}_\mathcal{O}(\sigma_{t+1, \mathcal{O}}^2) \), where \( \mathbb{E}_\mathcal{O} \) represents the average over \( \mathcal{O} \in \{ \text{BMO, BLO, SLO, SMO} \} \), with weights \( P_\mathcal{O} \). Then, \( \mathbb{E}_\mathcal{O}(\delta_{t+1, \mathcal{O}}) = \mathbb{E}_\mathcal{O}(V_{t+1, \mathcal{O}}) = 0 \), and

\[
\bar{\mu}_{t+1} = \mu - \mathbb{E}_\mathcal{O} \delta_{t+1, \mathcal{O}}, \quad \bar{\sigma}_{t+1}^2 = \sigma_t^2 + \bar{\sigma}_{t+1, \mathcal{O}}^2 - \mathbb{E}_\mathcal{O} \delta_{t+1, \mathcal{O}}. 
\] (A6)

(c) If \( \psi_t(\cdot) = \phi(\cdot; \mu_t, \sigma_t) \) is normal, let \( i_{\mathcal{O}} = (L_{\mathcal{O}}, H_{\mathcal{O}}), \ell_{\mathcal{O}} = \frac{L_{\mathcal{O}} - \mu_t}{\sigma_t}, \) \( h_{\mathcal{O}} = \frac{H_{\mathcal{O}} - \mu_t}{\sigma_t}. \)

Then,

\[
P_{\mathcal{O}} = \frac{1 - \rho}{4} + \rho \left( \Phi(h_{\mathcal{O}}) - \Phi(\ell_{\mathcal{O}}) \right),
\]

\[
\mu_{t+1, \mathcal{O}} = \mu - \delta_{t+1, \mathcal{O}}, \quad \sigma_{t+1, \mathcal{O}}^2 = \sigma_t^2(1 + V_{t+1, \mathcal{O}}) + \sigma_t^2 - \delta_{t+1, \mathcal{O}}^2, 
\] (A7)

\[
\delta_{t+1, \mathcal{O}} = \frac{\rho \sigma_t \phi(\ell_{\mathcal{O}} - \phi(h_{\mathcal{O}}))}{P_{\mathcal{O}}}, \quad V_{t+1, \mathcal{O}} = \frac{\rho (\ell_{\mathcal{O}} \phi(h_{\mathcal{O}}) - h_{\mathcal{O}} \phi(h_{\mathcal{O}}))}{P_{\mathcal{O}}}. 
\]

If we write \( \bar{\mu}_{t+1} = f(\mu_t) \), then

\[
f'(\mu_t) = 1 - \frac{\rho^2}{\beta \sigma_t} \left( 2 \gamma \phi \left( \frac{\mu_t}{\sigma_t} \right) + (1 - \gamma) \left( \phi \left( \frac{\alpha + \mu_t}{\sigma_t} \right) + \phi \left( \frac{\alpha - \mu_t}{\sigma_t} \right) \right) \right). 
\] (A8)

(d) If \( \psi_t(\cdot) \) is the standard normal density, with \( \mu_t = 0 \) and \( \sigma_t = 1 \), then for all \( \mathcal{O} \) at \( t \),

\[
P_{\mathcal{O}} = \frac{1}{4}, \quad \delta_{t+1, \mathcal{O}} = \delta_{t+1}, \quad \mu_{t+1, \mathcal{O}} = 0, \quad \sigma_{t+1} = 1. 
\] (A9)

Hence, the normalized density \( \psi_t \) has constant volatility.

Proof. Conditional on observing \( w_t = \frac{v_t - v_E}{\sigma_E} = z \), the probability of an order \( \mathcal{O} \) at \( t \) is
\[ P(O_t = O \mid w_t = z) = (1 - \rho) \frac{1}{4} + \rho 1_{z \in I_O}. \] Indeed, if the trader at \( t \) is uninformed (with probability \( 1 - \rho \)), he submits an order \( O \) with equal probability \( \frac{1}{4} \); if the trader at \( t \) is informed (with probability \( \rho \)), she submits an order \( O \) if and only if \( z \in I_O \). Integrating over \( z \), we obtain \( P_O = \frac{1 - \rho}{4} + \rho \int_{z \in I_O} \psi_t(z)dz \), which proves (A1).

We now compute the density of the normalized asset value at \( t + 1 \) after observing an order \( O \) at \( t \). Immediately after \( t \) the efficient price moves to \( w_{t+1}^E = w_t^E + \Delta O \), where \( \Delta O \in \{ \Delta, \gamma \Delta, -\gamma \Delta, -\Delta \} \). Since \( \frac{\Delta}{\sigma_E} = \frac{\rho}{\beta} \), note that \( \delta O = \frac{\Delta}{\sigma_E} = \{ \frac{\rho}{\beta}, \gamma \frac{\rho}{\beta}, -\gamma \frac{\rho}{\beta}, -\frac{\rho}{\beta} \} \). If \( z = w_t \) and \( \delta v = \frac{\nu_{t+1} - \nu_t}{\sigma_E} \), write \( x = w_{t+1} = \frac{\nu_{t+1} - (w_t^E + \Delta O)}{\sigma_E} = \delta v + z - \delta O \). But \( \delta v \) has a normal distribution given by \( \mathcal{N}(0, \sigma^2) \), hence \( P(w_{t+1} = x \mid O_t = O, w_t = z) = P(\delta v = x - z + \delta O) = \phi(x - z + \delta O; 0, \bar{\sigma}) = \phi(x - z - \delta O, \bar{\sigma}) \). Compute also \( P(w_{t+1} = x, O_t = O \mid w_t = z) = P(w_{t+1} = x \mid O_t = O, w_t = z) P(O_t = O \mid w_t = z) = \phi(x - z - \delta O, \bar{\sigma}) \left( \frac{1 - \rho}{4} + \rho 1_{z \in I_O} \right) \). Thus, the posterior density is \( \psi_{t+1, O}(x) = \int \frac{P(w_{t+1} = x, O_t = O \mid w_t = z) \psi_t(z)dz}{P(O_t = O \mid w_t = z)} = \int \frac{ (\frac{1 - \rho}{4} + \rho 1_{z \in I_O}) \phi(x - z - \delta O, \bar{\sigma}) \psi_t(z)dz}{P_O} \). This proves (A2).

To prove part (b), we start by computing as above \( P(w_t = z \mid O_t = O) = \frac{1 - \rho}{4} + \rho 1_{z \in I_O} \). Multiplying by \( z \) and integrating, we get \( E(w_t \mid \psi_t, O) = \int z \frac{1 - \rho}{4} + \rho 1_{z \in I_O} \psi_t(z)dz \), and by subtracting \( \mu_t = E(w_t \mid \psi_t) \) we get \( \delta_{t+1, O} = \int \frac{ (\frac{1 - \rho}{4} + \rho 1_{z \in I_O}) (z - \mu_t) \psi_t(z)dz}{P_O} \). But \( \int (z - \mu_t) \psi_t(z)dz = 0 \), hence \( \delta_{t+1, O} = \int \frac{ (\frac{1 - \rho}{4} + \rho 1_{z \in I_O}) (z - \mu_t) \psi_t(z)dz}{P_O} \), which proves (A3).

To compute the mean of \( \psi_{t+1, O}(x) \), we integrate the formula (A2) over \( x \), and obtain \( \mu_{t+1, O} = \int \frac{ (\frac{1 - \rho}{4} + \rho 1_{z \in I_O}) (z - \mu_t) \psi_t(z)dz}{P_O} \). This is similar to the formula we have proved for \( \delta_{t+1, O} \), except that \( \mu_t \) is replaced by \( \delta O \). We get \( \mu_{t+1, O} = \delta_{t+1, O} + \mu_t - \delta O \), which proves the first part of (A4).

For the second part of (A4), note that for any (not necessarily normal) distribution \( \psi \) with mean \( \mu \) and variance \( \sigma^2 \), \( \int (x + a)^2 \psi(x)dx = \sigma^2 + (\mu + a)^2 \). Then,

\[
\int (x - \mu + \delta O)^2 \psi_{t+1, O}(x)dx = \sigma_{t+1, O}^2 + (\mu_{t+1, O} - \mu_t + \delta O)^2 = \sigma_{t+1, O}^2 + \delta_{t+1, O}^2. \tag{A10}
\]

We also integrate directly \( \int (x - \mu_t + \delta O)^2 \psi_{t+1, O}(x)dx \) by replacing \( \psi_{t+1, O}(x) \) as in (A2).

Using the formula \( \int (x - \mu_t + \delta O)^2 \phi(x; z - \delta O, \bar{\sigma})dx = (z - \mu_t)^2 + \bar{\sigma}^2 \), we obtain

\[
\int (x - \mu_t + \delta O)^2 \psi_{t+1, O}(x)dx = \sigma^2 + \int \psi_t(z) \left( \frac{1 - \rho}{4} + \rho 1_{z \in I_O} \right)(z - \mu_t)^2 dz. \tag{A11}
\]

Putting together (A10) and (A11), we get the desired formula for \( \sigma_{t+1, O}^2 \). Equation (A5) follows directly from (A3) and (A4). Finally, proving \( E_O(\delta_{t+1, O}) = 0 \) and \( E_O(V_{t+1, O}) = 0 \)
is straightforward, which also implies equation (A6).

To prove part (c), first use (A1) to compute $P_\mathcal{O} = \frac{1-\rho}{4} + \rho(\Phi(h_\mathcal{O}) - \Phi(\ell_\mathcal{O}))$. To prove the formula for $\delta_{t+1,\mathcal{O}}$, make the change of variable $z' = \frac{z-\mu_t}{\sigma_t}$ and denote by $v'_\mathcal{O} = (\ell_\mathcal{O}, h_\mathcal{O})$. Then, $\delta_{t+1,\mathcal{O}} = \frac{\rho v'_\mathcal{O} \phi(z')dz}{P_\mathcal{O}} = \frac{\rho \sigma_t (\phi(\ell_\mathcal{O}) - \phi(h_\mathcal{O}))}{P_\mathcal{O}}$. A similar computation for $V_{t+1,\mathcal{O}}$ finishes the proof of (A7). Finally, $\bar{\mu}_{t+1} = f(\mu_t) = \mu_t - \sum_\mathcal{O} P_\mathcal{O} \delta_\mathcal{O} = \mu_t - \rho \sum_\mathcal{O} (\Phi(h_\mathcal{O}) - \Phi(\ell_\mathcal{O})) \delta_\mathcal{O}$. If we differentiate the endpoints of $v'_\mathcal{O}$ with respect to $\mu_t$, we get $-\frac{1}{\sigma_t}$ in all cases, hence $f'(\mu) = 1 - \rho \sum_\mathcal{O} (\phi(h_\mathcal{O}) - \phi(\ell_\mathcal{O})) \left(-\frac{1}{\sigma_t}\right) \delta_\mathcal{O}$. Using $\delta_\mathcal{O} \in \{\frac{\rho}{\beta}, \gamma_\mathcal{O}, -\gamma_\mathcal{O}, -\frac{\rho}{\beta}\}$, a straightforward calculation proves (A8).

To prove part (d), we substitute $\mu_t = 0$ and $\sigma_t = 1$ in the formulas above. We only prove the results for $\mathcal{O} = \text{BMO}$ and $\text{BLO}$, the proof for the other order types being symmetric. The probability of a BMO is $P_{\text{BMO}} = \frac{1-\rho}{4} + \rho \int_0^\infty \phi(z)dz = \frac{1-\rho}{4} + \rho \frac{1}{4} = \frac{1}{4}$. The probability of a BLO is $P_{\text{BLO}} = \frac{1-\rho}{4} + \rho \int_0^\alpha \phi(z)dz = \frac{1-\rho}{4} + \rho \frac{1}{4} = \frac{1}{4}$.

The normalized price impact of a BMO is $\delta_{t+1,\text{BMO}} = \frac{\rho \int_0^\infty \phi(z)dz}{P_{\text{BMO}}} = \frac{\rho \phi(\alpha)}{\rho \phi(\alpha) + 1/4} = \frac{\rho}{\beta} = \delta_{\text{BMO}}$. The normalized price impact of a BLO is $\delta_{t+1,\text{BLO}} = \frac{\rho \int_0^\alpha \phi(z)dz}{P_{\text{BLO}}} = \frac{\rho \phi(\alpha)}{\rho \phi(\alpha) + 1/4} = \frac{\phi(0) - \phi(\alpha)}{\rho \phi(\alpha) + 1/4} = \gamma_\mathcal{O} = \delta_{\text{BLO}}$. By symmetry, it follows that $\delta_{t+1,\mathcal{O}} = \delta_\mathcal{O}$ for all orders $\mathcal{O} \in \{\text{BMO, BLO, SLO, SMO}\}$.

We now compute $\mu_{t+1,\mathcal{O}} = \mu_t - \delta_\mathcal{O} + \delta_{t+1,\mathcal{O}} = \mu_t$. Also, $\bar{\sigma}_{t+1}^2 = E_\mathcal{O}(\sigma_{t+1,\mathcal{O}}^2) = E_\mathcal{O}(\sigma_{t}^2 + \bar{\sigma}^2 - \delta_\mathcal{O}^2)$. But $E_\mathcal{O}(\delta_\mathcal{O}^2) = \frac{1}{4}((\frac{\rho}{\beta})^2 + (\gamma_\mathcal{O})^2 + (-\gamma_\mathcal{O})^2 + (-\frac{\rho}{\beta})^2) = \bar{\sigma}^2$, hence $\bar{\sigma}_{t+1}^2 = \sigma_{t}^2 + \bar{\sigma}^2 - \delta_\mathcal{O}^2 = \sigma_{t}^2$, from which $\bar{\sigma}_{t+1} = \sigma_{t} = 1$. Thus, the posterior mean is equal to zero irrespective of the order $\mathcal{O}$ at $t$, while the posterior variance is equal to one on average. This means that the normalized density $N(0, 1)$ is steady-state.

In the next two lemmas, we describe the continuation payoff from submitting a BLO for either a patient speculator (Lemma A2), or for an uninformed patient natural buyer (Lemma A3), assuming that all investors follow their equilibrium strategies.

To state the next result, define the execution probability function $g_1(\rho, w)$ as the information function $g$ in Definition 1, but with $\mu(Q) = 1$. Numerically, we verify that $g_1 \equiv 1$ (see Result 1), but for the next lemma we do not need any particular expression for $g_1$.

**Lemma A2.** In the context of Theorem 1, consider an informed trader who submits a BLO at $t$ after observing the signal $w_t = \frac{w_{t-1} + w_t}{\sigma_\mathcal{E}}$. Then, if subsequently all traders follow
the equilibrium strategies, the continuation payoff of the informed trader is

\[ U_{\text{BLO}}^t = \frac{s}{2} g_1(\rho, w_t) + \sigma_E g(\rho, w_t). \]  

(A12)

Proof. We simplify notation and assume that the initial BLO is submitted at \( t = 0 \). Denote by \( Q \) the set of all execution sequences \( Q = (O_0 = \text{BLO}, O_1, \ldots, O_{T-1}, O_T = \text{SMO}) \) for the initial BLO. Let \( J_t \) be the information set of the PS just before trading at \( t \), which consists of the signal \( w_0 \) observed at \( t = 0 \), and the orders \( O_0, \ldots, O_{t-1} \). Let \( E_t \) be the expectation operator conditional on \( J_t \). At the execution time \( T \), the bid price is \( v_T^E - s/2 \), therefore

\[ U_{\text{BLO}}^t = \sum_{Q \in Q} E_0 \left( v_T - \left( v_T^E - \frac{s}{2} \right) | Q \right) P_0(Q) \]

\[ = \frac{s}{2} \sum_{Q \in Q} P_0(Q) + \sum_{Q \in Q} I(Q) P_0(Q), \]  

(A13)

where \( I(Q) = E_0 \left( v_T - v_T^E | Q \right) \).

For \( t = 1, \ldots, T+1 \), let \( P_t \) be the probability of observing the order \( O_t \) at \( t \) conditional on \( J_t \), \( \psi_t \) the density of \( w_t \) before trading at \( t \), and \( \mu_t = E_t(w_t) \) the mean of \( \psi_t \). We show that the sequence of probabilities \( (P_1, \ldots, P_T) \), densities \( (\psi_1, \ldots, \psi_{T+1}) \), and means \( (\mu_1, \ldots, \mu_{T+1}) \) is indeed associated to the execution sequence \( Q \), in the sense of Definition 1. From equations (A1) and (A2) in Lemma A1, it follows that \( P_t = \pi_{\psi_t, O_t} \) and \( \psi_{t+1} = f_{\psi_t, O_t} \), where \( \pi \) and \( f \) are given by equation (3) in Definition 1.

Next, we show that \( P_0(Q) \) coincides with \( P(Q) = \prod_{t=1}^T P_t \) from Definition 1. Indeed, \( P_0(Q) = P(O_1, \ldots, O_T | O_0) = \prod_{t=1}^T P(O_t | O_{0}, \ldots, O_{t-1}) = \prod_{t=1}^T P_t = P(Q) \). Since by definition \( g_1(\rho, w_0) = \sum_{Q \in Q} P(Q) \), using (A13) we obtain the first half of (A12).

We are left to prove that \( \sum_{Q \in Q} I(Q) P(Q) = \sigma_E g(\rho, w_0) \). First, we show that \( \psi_1 = N(0, \frac{\gamma_0}{\beta}, \frac{1+\gamma_2}{2\beta^2}) \), as specified in the definition of \( g \). To see this, note that \( \delta_{O_0} = \delta_{\text{BLO}} = \gamma_0/\beta \). Then, \( w_1 = w_0 + \frac{\nu_1 - \nu_0}{\sigma_E} - \frac{v_T^E - v_T^E}{\sigma_E} = w_0 + \frac{\nu_1 - \nu_0}{\sigma_E} - \gamma_0/\beta \). Because \( v_1 - v_0 \sim N(0, \sigma^2) \), we have \( \text{Var} \left( \frac{\nu_1 - \nu_0}{\sigma_E} \right) = \frac{\sigma^2}{\sigma_E^2} = \rho^2 \frac{1+\gamma_2}{2\beta^2} \), where the last equality follows from (13). Since \( \psi_1 \) is the density of \( w_1 \), we obtain indeed \( \psi_1 = N \left( w_0 - \gamma_0/\beta, \rho^2 \frac{1+\gamma_2}{2\beta^2} \right) \).

Finally, we show that \( I(Q) = \sigma_E \mu(Q) \), where \( \mu(Q) = \mu_{T+1} - \frac{\rho}{\beta} \). The executing order at \( T \) is a SMO, therefore (A5) implies that \( E_T(w_T | \text{SMO}_T) = \mu_{T+1} + \delta_{\text{SMO}} = \mu_{T+1} - \frac{\rho}{\beta} \). Thus, \( I(Q) = \sigma_E E_0 E_T(w_T | \text{SMO}_T) = \sigma_E (\mu_{T+1} - \frac{\rho}{\beta}) \).

For future reference, we note that according to (A5) we have the following decom-
\[
\mu_{T+1} - \frac{\rho}{\beta} = \mu_{T+1} + \delta_{\text{SMO}_T} = \mu_T + \delta_{T+1,\text{SMO}_T}, \tag{A14}
\]

where, as shown in (A3), \(\delta_{T+1,\text{SMO}_T}\) is the normalized adverse selection from \(\text{SMO}_T\).

**Lemma A3.** In the context of Theorem 1, consider a patient uninformed trader with private valuation \(\bar{u}\), who submits a BLO at \(t\). Then, if subsequently all traders follow the equilibrium strategies, the continuation payoff of the uninformed trader is

\[
U_{\text{BLO}}^U = \bar{u} + \frac{s}{2} - \Delta. \tag{A15}
\]

**Proof.** We simplify notation and assume that the initial BLO is submitted at \(t = 0\). In Section 2, we assume that the initial belief of an uninformed trader with private valuation \(\bar{u}\) is such that after submitting a BLO at \(t = 0\), his posterior belief at \(t = 1\) is the efficient density, \(v_1^E = \mathcal{N}(v_1^E, \sigma_E^2)\). Formally, this is done by assuming that before trading at \(t = 0\), the uninformed trader believes \(v_0\) to be distributed according to \(\mathcal{N}(v_0^E + \gamma \Delta, \sigma_E^2 - \sigma^2)\). By submitting his BLO at \(t = 0\), he instantly affects the efficient price according to \(v_1^E = v_1^E + \gamma \Delta\). His belief about \(v_0\), however, stays the same, since he knows he is uninformed. The asset value evolves according to \(v_1^E = v_0^E + (v_1^E - v_0^E)\), with the increment normally distributed according to \(\mathcal{N}(0, \sigma^2)\). Therefore, at \(t = 1\) he believes \(v_1 \sim \mathcal{N}(v_1^E, (\sigma_E^2 - \sigma^2) + \sigma^2)\), which is the efficient density at \(t = 1\).

As in Lemma A2, we denote by \(Q\) the set of all execution sequences \(Q = (O_0 = \text{BLO}, O_1, \ldots, O_{T-1}, O_T = \text{SMO})\) for the initial BLO. At the execution time \(T\), the bid price is \(v_T^E - s/2\), therefore

\[
U_{\text{BLO}}^U = \bar{u} + \frac{s}{2} \sum_{Q \in Q} P_0(Q) + \sum_{Q \in Q} E_0(v_T - v_T^E | Q) P_0(Q), \tag{A16}
\]

where \(E_0\) is the expectation operator conditional on \(I_0\), the public information set just before trading at \(t\); and \(P_0(Q)\) is the probability that the sequence \(Q\) will occur conditional on \(I_0\). Denote by \(I(Q) = E_0(v_T - v_T^E | Q)\). The executing order at \(T\) is a SMO, therefore \(I(Q) = E_0 E_T(v_T - v_T^E | \text{SMO}_T)\). Equation (A3) implies \(E_T(w_T | \text{SMO}_T) = E_T(w_T + \delta_{\text{SMO}_T}) = E_T\left(\frac{(v_T - v_T^E)}{\sigma_E}\right) - \frac{\Delta}{\sigma_E}\). Thus, \(I(Q) = E_0 E_T(v_T - v_T^E) - \Delta\). But \(E_T(v_T) = v_T^E\), therefore \(I(Q) = -\Delta\).

To finish the proof, we now show that \(\sum_{Q \in Q} P_0(Q) = 1\), which means that the initial

\[^{37}\text{Equation (13) implies that } \sigma_E^2 - \sigma^2 > 0.\]
BLO is executed with probability one. We use the theory of absorbing Markov chains, as described in Chapter 11 of Grinstead and Snell (2003). We briefly indicate the proof. Consider the following Markov chain with a countable number of states, where the state $j \geq 0$ indicates that the initial BLO is $j$th in the bid queue. The absorbing state $j = 0$ means that the BLO is executed. Then, from each state $j \geq 1$ the system moves to either $j - 1$ with probability $1/4$ (if SMO occurs), to $j + 1$ with probability $1/4$ (if BLO occurs), or remains in $j$ with probability $1/2$ (if either BMO or SLO occurs). One can then check that the fundamental matrix $M$ (corresponding to the non-absorbing states) has entry $(i, j)$ given by $M_{ij} = 4 \min \{i, j\}$, for $i, j \geq 1$. The matrix of transition to the absorbing state is the column matrix $R$, whose $j$th entry is either $1/4$ if $j = 1$, or 0 if $j > 1$. Theorem 11.6 from Grinstead and Snell (2003) shows that the probability of absorption (execution) starting from state $j$ is the $j$th entry in the column matrix $B = MR$. But $B$ has all entries equal to one. This completes the proof of (A15).

We finish the discussion by analyzing the payoff of the uninformed trader in state $j \geq 0$, when the BLO is $j$th in the bid queue. By using the same argument as above, it follows that his payoff is affected only by the amount of adverse selection at the time $T$ when his order is executed. Thus, his payoff is the same regardless of $j$. This is not surprising, since the uninformed trader has a zero waiting cost. 

**Verification of Result 1.** See Internet Appendix Section 3.

**Proof of Theorem 1.** The proof of the theorem depends on the conditions in Result 1 being analytically true. We thus assume that, for all $\rho \in (0, 1),$

$$g(\rho, w), w - g(\rho, w), \text{ and } g(\rho, w) - g(\rho, -w) \text{ are strictly increasing in } w,$$

$$\max \left( \frac{\rho(1 + \gamma)}{\beta}, -2g(\rho, 0) - 2\frac{\rho \gamma}{\beta} \right) < \alpha - g(\rho, \alpha),$$

$g = g(\rho, \psi_1, j)$ increases if the density $\psi_1$ has a positive shift in mean,

$g = g(\rho, w, j)$ decreases in $j$ if $w > 0$, and

$g_1(\rho, w, j) = 1$ for all $w$ and $j \geq 1$,

(A17)

where $g = g(\rho, w, j)$ is as in Definition 1 but for a more general integer $j = j_0 \geq 1$; $g = g(\rho, \psi_1, j)$ is as in Definition 1 but for a more general density $\psi_1$ and integer $j = j_0 \geq 1$; also, $g_1(\rho, w)$ is as in Definition 1 but for a more general integer $j = j_0 \geq 1$ and $\mu(Q) = 1$ in (4).
We now summarize the trading game described in Section 2. At each integer time \( t = 1, 2, \ldots \) (corresponding to clock times \( \frac{1}{\lambda}, \frac{2}{\lambda}, \ldots \)) the asset value changes according to

\[ v_t = v_{t-1} + \sigma \varepsilon_t, \]

where \( \sigma = \frac{\sigma_E}{\sqrt{\lambda}} \) is the inter-trade volatility, and \( \varepsilon_t \sim \mathcal{N}(0, 1) \). At each integer time \( t \geq 0 \), a player called “Nature” draws a new trader which is (i) informed with probability \( \rho \), meaning that she observes \( v_t \), or (ii) uninformed with probability \( 1 - \rho \), in which case with equal probability the trader is either a patient natural buyer (with private valuation \( \bar{u} \)), patient natural seller (with private valuation \( -\bar{u} \)), impatient natural buyer (who always submits a BMO), or impatient natural buyer (who always submits a SMO).\(^{38}\) A trader that arrives to the market at \( t \) either chooses an order of type \{BMO, BLO, SLO, SMO\} for one unit of the asset, or submits no order (NO). In the latter case, the trader exits the model forever, and Nature immediately draws another trader from the pool. At non-integer times the game is played with the existing traders in the limit order book. The game is set in continuous time, based on the framework of Bergin and MacLeod (1993). Thus, the game allows for instantaneous responses, by completing the space of strategies with respect to the response time.

We now define a Markov Perfect Equilibrium (MPE) of the game. Since the MPE is a refinement of the notion of Perfect Bayesian Equilibrium (Fudenberg and Tirole 1991, Section 13.2), we define a strategy profile and a belief system that are compatible with each other, in the sense that, at every stage of the game, strategies are optimal given the beliefs, and the beliefs are obtained from equilibrium strategies and observed actions using Bayes’ rule. The MPE concept simplifies the description of the game by specifying a set of state variables that summarizes the payoff-relevant information contained in each history of the game. In our case, the state variables are

- Public variables: the efficient density, given by the efficient price \((v^E_t)\) and the efficient volatility \((\sigma^E_t)\); and the limit order book, given by the bid price \((b_t)\), the ask price \((a_t)\), and the bid and ask queues.\(^{39}\)

- Private variable for informed traders at the time of arrival: the asset value \((v_t)\).

Because we want the game to be already in steady-state, we assume that an instant before \( t = 0 \) the efficient price is \( v^E_0 = 0 \) and the efficient volatility is the parameter \( \sigma_E \) from (5), so that the initial efficient density is \( \mathcal{N}(0, \sigma^2_E) \). The ask price is \( v^E_0 + s/2 \), the

\(^{38}\)The decisions of the impatient traders are endogenized in the Internet Appendix Section 1.

\(^{39}\)Because in our model traders can submit orders only for one unit, the limit prices for orders other than the first ones in the bid and ask queues are not relevant.
bid price is $v^E_0 - s/2$, where $s$ is the parameter from (5), while the initial limit order book has countably many limit orders on each side (see the middle plot in Figure 2).

To simplify the description of the strategies, we used the one-stage deviation principle of subgame perfection (see Fudenberg and Tirole 1991, Section 4.2).\textsuperscript{40} This principle implies that we do not need to define the strategy profile for all conceivable histories, but only for the histories that arise from at most a finite number of (out-of-equilibrium) deviations, assuming that all other players than the transgressor act according to their strategies at the time of the deviation. Using this principle, we restrict the values that certain state variables can take.

\textbf{Remark A1.} By the one-stage deviation principle of subgame perfection, we assume that at all times (integer or not) the efficient volatility is equal to the parameter $\sigma_E$.\textsuperscript{41} At all integer times $t = 0, 1, \ldots$, the ask price is $v^E_t + s/2$ and the bid price is $v^E_t - s/2$.\textsuperscript{42}

To define the strategy profile $S$, we first describe the action of a new trader who arrives at $t$. Then, we describe the reaction of the other traders remaining in the limit order book to the new arrival at $t$. Finally, we describe the reaction of the existing traders to any out-of-equilibrium deviation that might occur from either the new trader or an existing trader. Recall that impatient traders are assumed to automatically submit market orders. We therefore describe only the strategies of patient traders, who can be informed (with private valuation 0), uninformed buyers (with private valuation $\bar{u}$), or uninformed sellers (with private valuation $-\bar{u}$). The strategy profile $S$ is then given by the following set of rules:

(a) The uninformed buyer arriving at $t$ submits a BLO at the price $(v^E_t + \gamma \Delta) - s/2$.

(b) The uninformed buyer arriving at $t$ submits a SLO at the price $(v^E_t - \gamma \Delta) + s/2$.

(c) The informed trader who observes an asset value $v_t$ when she arrives at $t$ submits an order $O \in \{BMO, BLO, SLO, SMO\}$ whenever her signal $\frac{v_t - v^E_t}{\sigma_E}$ lies, respectively, in the interval $\{(\alpha, \infty), (0, \alpha), (-\alpha, 0), (-\infty, -\alpha)\}$.

\textsuperscript{40}The principle states that in order to verify that an equilibrium is subgame perfect, it suffices to check whether there is any history $h'$ where some player $i$ can gain by deviating from the action prescribed by his strategy at $h'$ and conforming to it thereafter—assuming all other players follow their strategies.

\textsuperscript{41}At $t = 0$ this is true by construction. Later, this is true both in equilibrium (see part (d) of Lemma A1), and out-of-equilibrium, since the efficient density after any deviation remains equal to $\sigma_E$ (see $S(g)$ and $S(h)$).

\textsuperscript{42}Indeed, even after deviations that modify the bid-ask spread, the instant reactions of the other traders would restore the bid and ask prices at the correct values (see $S(g)$).
(d) After the initial order submission, all traders behave as described in (e) and (f).

(e) If a BMO is submitted at $t$, then an instant later the efficient price is updated to $v_t^E + \Delta$, the ask price to $v_t^E + \Delta + s/2$, and the bid price to $v_t^E + \Delta - s/2$, and all other limit traders shift their orders by $\Delta$ such that the relative ranks in the ask and bid queues are preserved. After that, no other trader moves until $t + 1$.

The reaction to a SMO at $t$ is symmetric to the reaction to a BMO.

(f) If a BLO is submitted at $t$, then an instant later the efficient price is updated to $v_t^E + \gamma \Delta$, the ask price to $v_t^E + \gamma \Delta + s/2$, and the bid price to $v_t^E + \gamma \Delta - s/2$, and all other limit traders shift their orders by $\gamma \Delta$ such that the relative ranks in the ask and bid queues are preserved. After that, no other trader moves until $t + 1$.

The reaction to a SLO at $t$ is symmetric to the reaction to a BLO.

(g) (Out-of-equilibrium behavior) If an uninformed trader submits a limit order of different type than specified by his equilibrium strategy, then immediately he switches to a limit order of the type specified in (a) and (b). For any such limit order switch, the other traders believe the transgressor is uninformed with probability one. If a trader submits a limit order at a price not specified by his equilibrium strategy, thereafter he behaves as described in (e), (f), and (h).

(h) (Out-of-equilibrium behavior) If a limit order trader on the bid side deviates from the behavior above and instead of $b^*$ submits an order at $b = b^* + d$, then

- If $d < 0$, the other traders believe the transgressor is uninformed with probability one, and no state variables change, unless the transgressor is the first in the bid queue, in which case another limit buyer modifies his BLO at $b$.
- If $d > 0$, an instant later the other traders believe the transgressor is informed with probability one. As a result, the transgressor’s private information is revealed, and the efficient price is updated to $v_t^E + d'$, with $d' > d$, while the efficient volatility remains $\sigma^E$.\footnote{We also require that $d'$ is a one-to-one function of $d$. The exact value of $d' > d$ is not important, but because the transgressor gets price priority at $b = b^* + d$, some limit orders need to be shifted by more than $d$ to preserve the relative ranks in the bid and ask queue. An out-of-equilibrium belief can be arbitrary, but it should be about the transgressor’s type rather than directly about the resulting public belief on $v$. A belief about an investor’s type is a pair $(p, \psi)$, where $p$ is a probability that the transgressor is informed, and $\psi$ is the density for the initial $v$. Since the updating function $f_{\psi, \rho}$ in equation (3) in the paper can be shown to be one-to-one for densities, it is sufficient to specify directly}
of the book shifts up by either $d$ or $d'$, such that the relative ranks in the bid and ask queues are the same as before the deviation.

The case when the transgressor is on the ask side is symmetric.

The belief system is described by the following rules: At $t = 0$, the uninformed investors perceive the asset value distributed according to $\mathcal{N}(0, \sigma_E^2)$. Subsequently, the uninformed investors’ belief about the asset value (the efficient density) is updated using the approximate Bayes’ rule described in Section 2. At the time of arrival to the market, the informed investor observes the asset value and can compute the average payoff of a limit order based on updating her belief according to the exact Bayes’ rule. After the arrival, however, the informed trader cannot update her belief, and becomes essentially uninformed. In the limit order book at $t = 0$, all traders are uninformed with probability one. At $t \geq 0$, each new trader is believed to be informed with probability $\rho$ by the other traders. Subsequently, traders’ beliefs about the other investors’ types are updated according to the Bayes’ rule.

Because the strategy profile $S$ defined above depends only on the current value of the state variables, the strategies are indeed Markov. We now show that the strategy of each type of investor is a best response to the other investors’ strategies.

**Uninformed Traders**

We analyze only the strategy of an uninformed buyer, since the proof for an uninformed seller is symmetric. We thus need to prove that the strategy of an uninformed buyer described in $S(a)$, $S(e)$–$S(h)$ is optimal. Consider a (patient) uninformed buyer, with private valuation $\bar{u} > s/2$. If he arrives at $t$, we first analyze his choice among one of the following options: (i) BMO, (ii) SMO, (iii) NO (no order), (iv) BLO at $b^* = (v_E - d' + \gamma \Delta) - s/2$, and later follow $S$, or (v) SLO at $a^* = (v_E - \gamma \Delta) + s/2$, and later follow $S$. Option (v) can be ruled out, because the $S(g)$ requires that the uninformed buyer immediately reverses his SLO to a BLO. But the other traders believe with probability one that the transgressor is uninformed, and hence the outcome of option (v) is equivalent to option (iv).

---

The belief of the transgressor at $t$: $v_t \sim \mathcal{N}(\mu_t, \sigma_t^2)$, with $\mu_t = v_E + d', \sigma_t^2 = (\sigma_E^2 - \sigma^2_0)$. (Note that equation (13) implies $\sigma_E > \sigma$.) Since $d$ is observed by the public and $d'$ is a one-to-one function of $d$, the transgressor’s private information is fully revealed to the public. Thus, the public at $t$ also has the belief $\mathcal{N}(\mu_t, \sigma_t^2)$. Because the asset value has a normal increment with density $\mathcal{N}(0, \sigma^2)$, the efficient density at $t + 1$ becomes $\mathcal{N}(v_E + d', \sigma^2_0)$.
We first show that, as specified by $S$, the option (iv) dominates all the other options when the limit prices in (iv) have the equilibrium values, $b^*$ and $a^*$, respectively. Then, we show that option (iv) is less profitable when the limit prices have different values. Moreover, we show that option (v) is less profitable when the SLO is switched to BLO after a lag—the proof presented below only works for an infinite lag (the SLO is never switched to BLO), but the argument is similar for a finite lag. In conclusion, the uninformed buyer always submits a BLO at the equilibrium price, which proves that $S(a)$ is optimal. Moreover, we rule out subsequent one-stage deviations after (iv) or (v) are chosen, which proves that $S(e)$ and $S(f)$ are optimal for the uninformed trader (as well as $S(g)$). Finally, if another trader later deviates by submitting a limit order at a non-equilibrium price, then the uninformed buyer optimally reacts as specified in $S(h)$.

Let $U^U$ be the expected payoff from submitting $O \in \{BMO, BLO, NO, SLO, SMO\}$ and later following $S$, except that in the case of $O^U_{SLO}$ we assume that the SLO is not switched to BLO. As explained in Remark A1, we assume that the current limit order book is such that the ask price is $a_t = v^E_t + s/2$, and the bid price is $b_t = v^E_t - s/2$. We show that $U^U_{BLO} > U^U_{BMO} > U^U_{NO} > U^U_{SLO} > U^U_{SMO}$. Recall that an uninformed buyer who arrives at $t$ believes the asset value $v_t$ to be distributed according to $N(v^E_0 + \gamma \Delta, \sigma^2)$ (see the proof of Lemma A3). Then, Lemma A3 shows that $U^U_{BLO} = s/2 - \Delta + \bar{u}$, and also rules out SLO, since this has a lower payoff than BLO.

We also compute $U^U_{BMO} = E_t(v_t) - a_t + \bar{u} = (v^E_t + \gamma \Delta) - (v^E_t + s/2) + \bar{u} = \bar{u} + \gamma \Delta - s/2$, $U^U_{SMO} = b_t - E_t(v_t) - \bar{u} = (v^E_t - s/2) - (v^E_t + \gamma \Delta) - \bar{u} = -s/2 - \gamma \Delta - \bar{u}$, and $U^U_{NO} = 0$. Collecting these formulas, we get

$$U^U_{BMO} = \bar{u} + \gamma \Delta - \frac{s}{2}, \quad U^U_{SMO} = -\bar{u} - \gamma \Delta - \frac{s}{2}, \quad U^U_{NO} = 0. \quad (A18)$$

By inspection, SMO and NO can be ruled out, because BMO clearly yields a larger payoff ($\bar{u} > s/2$). To rule out BMO, we note that condition (A17) implies $\alpha - g(\rho, \alpha) > \frac{\rho}{\beta}(1+\gamma)$, which if we multiply by $\sigma_E = \beta \rho^{-1} \Delta$ implies $s > \Delta(1+\gamma)$ and hence $U^U_{BMO} < U^U_{BLO}$.

We now show that the continuation payoff for the uninformed buyer from submitting or maintaining his BLO at the equilibrium price $b^*$ is at least as large as the payoff obtained by choosing BLO at either $b > b^*$ or $b < b^*$.

(D1) We first rule out BLO at $b > b^*$. According to $S(h)$, overshooting a bid leads to a shift in the efficient price by a positive quantity. But the trader knows that he
is in fact uninformed and that the correct efficient price is lower. This amounts to getting a negative shift in the mean of his belief about the fundamental value. Condition (A17) then implies that a negative shift in mean for the trader’s density brings a decrease in expected payoff.

(D2) We also rule out BLO at \( b < b^* \). According to \( S(h) \), undershooting a bid at the bid price does not change the efficient price, but if the transgressor is the first in the bid queue, it prompts another trader in the bid queue to immediately modify his BLO at \( b^* \). Then, the transgressor loses his first rank in the bid queue, which according to Lemma A3 does not change his payoff.\(^{44}\)

Next, we prove the optimality of \( S(h) \), which describes the response of the uninformed buyer to another trader who deviates from the equilibrium by choosing a limit order at \( b = b^* + d \) instead of \( b^* \). Note that according to \( S(h) \), the out-of-equilibrium belief specifies that the efficient price moves up by a quantity \( d' > d \). Then, the uninformed buyer finds himself in the same situation as before, when he had to decide whether to submit his bid at the equilibrium price or not. A similar argument as in the cases (D1) and (D2) above shows that it is indeed optimal for the uninformed buyer to also shift his bid by the amount specified by \( S(h) \).

**Informed Traders**

We prove that the strategy of an informed trader described in \( S(c), S(e), S(f), \) and \( S(g) \) is optimal. Consider a (patient) informed trader who arrives at \( t \) and observes the asset value \( v_t \), or equivalently the signal \( w_t = \frac{v_t - v_E^t}{\sigma_E} \). The informed trader has the option to submit either (i) BMO, (ii) SMO, (iii) NO (no order), (iv) BLO at \( b^* = (v_E^t + \gamma \Delta) - s/2 \), and later follow \( S \), (v) SLO at \( a^* = (v_E^t - \gamma \Delta) + s/2 \), and later follow \( S \). We show that the informed trader submits \( O \in \{ \text{SMO}, \text{SLO}, \text{BLO}, \text{BMO} \} \) whenever \( w_t \) lies, respectively, in the interval \( \{(-\infty, -\alpha), (-\alpha, 0), (0, \alpha), (\alpha, \infty)\} \); for this, we show that option (iii) is eliminated by a penalty for not trading that satisfies \( \nu \geq \gamma \Delta \). Then, if \( w_t > 0 \), we show that option (iv) is less profitable if the BLO is submitted at a different price than \( b^* \); symmetrically, if \( w_t < 0 \), we show that option (v) is less profitable if the SLO is submitted at a different price than \( a^* \). After submitting (iv) or (v), the informed trader

\(^{44}\)If we introduce an infinitesimal waiting coefficient, however, the transgressor is worse off, since with a higher rank in the queue he must wait longer to execute his order.
behaves in the same way as the uninformed buyer, which implies that the equilibrium is pooling.

Let \( \mathcal{U}_O^l \) be the continuation payoff from submitting \( O \) and later following \( S \). As in the case of the uninformed buyer, we assume that the current limit order book has the ask price \( a_t = v_t^E + s/2 \), and the bid price \( b_t = v_t^E - s/2 \). Let \( \tilde{\mathcal{U}}_O^l = \frac{\mathcal{U}_O^l}{\sigma_E} \) be the normalized payoff from \( O \); \( \hat{s} = \frac{s}{\sigma_E} \) the normalized bid-ask spread parameter; and \( \tilde{\nu} = \frac{\nu}{\sigma_E} \) the normalized commitment parameter, which is a penalty for non-trading. From Lemma A2, \( \tilde{\mathcal{U}}_{BLO}^l = \frac{\hat{s}}{2} g_1(\rho, w_t) + g(\rho, w_t) \). But, by condition (A17), \( g_1(\rho, w_t) \equiv 1 \), hence \( \tilde{\mathcal{U}}_{BLO}^l = \frac{\hat{s}}{2} + g(\rho, w_t) \). Putting together all the formulas, we get:

\[
\begin{align*}
\tilde{\mathcal{U}}_{BMO}^l &= w_t - \frac{\hat{s}}{2}, \\
\tilde{\mathcal{U}}_{SMO}^l &= -\frac{\hat{s}}{2} - w_t, \\
\tilde{\mathcal{U}}_{NO}^l &= -\tilde{\nu}, \\
\tilde{\mathcal{U}}_{BLO}^l &= \frac{\hat{s}}{2} + g(\rho, w_t), \\
\tilde{\mathcal{U}}_{SLO}^l &= \frac{\hat{s}}{2} + g(\rho, -w_t).
\end{align*}
\]

Denote by \( A(w) = w - g(\rho, w) \), \( B(w) = w - g(\rho, -w) \), \( D(w) = g(\rho, w) - g(\rho, -w) \); and note that \( B(w) = A(w) + D(w) \). With these notations, we get:

\[
\begin{align*}
\tilde{\mathcal{U}}_{BMO}^l - \tilde{\mathcal{U}}_{BLO}^l &= A(w_t) - \hat{s}, \\
\tilde{\mathcal{U}}_{BMO}^l - \tilde{\mathcal{U}}_{SLO}^l &= B(w_t) - \hat{s}, \\
\tilde{\mathcal{U}}_{BLO}^l - \tilde{\mathcal{U}}_{SMO}^l &= \hat{s} - B(-w_t), \\
\tilde{\mathcal{U}}_{SLO}^l - \tilde{\mathcal{U}}_{SMO}^l &= \hat{s} - A(-w_t).
\end{align*}
\]

From (A17), it follows that that \( A, D, \) and \( B = A + D \) are strictly increasing in \( w_t \), therefore all the payoff differences above are strictly increasing in \( w_t \). Note that by the definition of \( s \), we have \( A(\alpha) = \alpha - g(\rho, \alpha) = \hat{s} \), therefore BMO is preferred to BLO if and only if \( w_t > \alpha \). Similarly, SMO is preferred to SLO if and only if \( w_t < -\alpha \). Also, \( D(0) = 0 \), therefore BLO is preferred to SLO if and only if \( w_t > 0 \). Because all the payoff differences are strictly increasing in \( w_t \), a straightforward analysis shows that indeed the informed trader prefers \( O \in \{BMO, BLO, SLO, SMO\} \) whenever \( w_t \) lies, respectively, in the interval \( \{ (\alpha, \infty), (0, \alpha), (-\alpha, 0), (-\infty, -\alpha) \} \).

Next, we make sure that NO (“No Order”) is never optimal. For that, we use equation (A19) to compute the minimum payoff for each type of order. According to condition (A17), \( g \) is strictly increasing in \( w_t \), therefore \( \tilde{\mathcal{U}}_{O}^l \) is increasing in \( w_t \) for BMO and BLO, and decreasing in \( w_t \) for SMO and SLO. Thus, it is sufficient to verify that \( \tilde{\mathcal{U}}_{BLO}^l \geq -\tilde{\nu} \) when \( w_t = 0 \). Since by assumption \( \nu \geq \gamma \Delta \), the formula \( \sigma_E = \beta \rho^{-1} \Delta \) implies \( \tilde{\nu} \geq \frac{\rho \gamma}{\beta} \). Hence, it is sufficient to verify that \( \frac{\hat{s}}{2} + g(\rho, 0) \geq -\frac{\rho \gamma}{\beta} \). But this follows from condition (A17), which implies that \( \alpha - g(\rho, \alpha) > -2g(\rho, 0) - 2\frac{\rho \gamma}{\beta} \).

We now show that the continuation payoff for the informed trader from submitting his BLO at the equilibrium price \( b^* \) is higher than the payoff obtained by choosing BLO
at either \( b > b^* \) or \( b < b^* \). We rule out BLO at \( b > b^* \) by the same argument as in (D1) for the uninformed trader. We also rule out BLO at \( b < b^* \), because by \( S(h) \) this deviation prompts another trader in the bid queue to immediately modify his BLO at \( b^* \). The informed trader thus loses his first rank in the bid queue, which according to Lemma A2 generates a normalized continuation payoff of \( \tilde{s} + g(\rho, w_t, j) \), where \( j > 1 \) is the informed trader’s new rank in the bid queue. But by condition (A17), \( g(\rho, w_t, j) \) is decreasing in \( j \), which implies \( \tilde{s} + g(\rho, w_t, j) < \tilde{s} + g(\rho, w_t, j = 1) = \tilde{U}_{BLO}^I \). Hence, the informed trader reduces his payoff by deviating from \( b = b^* \).

Finally, after the initial order choice the strategy of the informed trader is the same as for the uninformed trader, since they now have the same information set.

Proof of Corollary 1. The corollary follows directly from the description of the equilibrium strategy profile \( S \), and in particular from \( S(e) \) and \( S(f) \).

Proof of Corollary 2. This corollary follows directly from the description of the equilibrium strategy profile \( S \), and in particular from \( S(c) \). The formula for the expected utility of the informed trader follows from equation (A19).

Proof of Corollary 3. As proved in Theorem 1, the efficient density at \( t \) is \( \mathcal{N}(v_t^E, \sigma_E^2) \), which implies that the normalized efficient density (the density of the signal \( w_t = \frac{v_t - v_E}{\sigma_E} \)) is standard normal. Then, part (d) of Lemma A1) shows that all orders have probability equal to \( 1/4 \).

Proof of Corollary 4. This corollary follows directly from the description of the equilibrium strategy profile \( S \), and in particular from \( S(a) \) and \( S(b) \). The formula for the expected utility of the uninformed trader follows from Lemma A3.

Proof of Proposition 1. From Corollary 1, any order \( O \in \{BMO, BLO, SLO, SMO\} \) moves the efficient price \( v_t^E \) by \( \Delta_O \in \{\Delta, \gamma \Delta, -\gamma \Delta, -\Delta\} \), respectively. Because each type of order occurs with probability \( \frac{1}{4} \), and the efficient price moves by \( \{\Delta, \gamma \Delta, -\gamma \Delta, -\Delta\} \), it is simple to show that the variance of \( v_{t+1}^E - v_t^E \) is indeed equal to \( \frac{1+\gamma^2}{2} \Delta^2 \).

Proof of Corollary 5. By Corollary 1, if the efficient price is \( v^E \), at any time the ask price is \( v^E + s/2 \), and the bid price is \( v^E - s/2 \). This implies that the bid-ask spread is equal to the parameter \( s = (\alpha - g(\rho, \alpha)) \sigma_E \) from (5), and is therefore constant.  

\[ \text{By condition (A17), } g_t \equiv 1. \]
Proof of Proposition 2. The proposition follows from the proof of Lemma A2 in the Appendix.

Proof of Corollary 6. By equation (5), \( s = (\alpha - g(\rho, \alpha)) \sigma_E = \text{Decay Cost}_\alpha. \)

Proof of Proposition 3. Recall that the slippage function \( g^s \) follows Definition 1, except that \( \mu(Q) = \mu_T \) instead of \( \mu(Q) = \mu_{T+1} - \frac{\rho}{\beta} \) for the information function \( g \). This proves the formula \( g^s(\rho, w) = E^c E_T(w_T) \). The adverse selection function \( g - g^s = g^a \) therefore follows Definition 1, except that \( \mu(Q) = (\mu_{T+1} - \frac{\rho}{\beta}) - \mu_T \). But equation (A14) implies \( \mu_{T+1} - \frac{\rho}{\beta} = \mu_T + \delta_{T+1,\text{SMO}_T} \). Hence, \( g^a \) is defined using \( \mu(Q) = \delta_{T+1,\text{SMO}_T} \), which is the price impact of the SMO at execution time \( T \). But this is equal to \( E_{T+1}(w_T) - E_T(w_T) \), which proves that indeed \( g^a(\rho, w) = E^c(E_{T+1}(w_T) - E_T(w_T)) \). 

Proof of Corollary 7. By equation (20), the slippage component satisfies \( s^s = (\alpha - g^s(\rho, \alpha)) \sigma_E = \text{Slippage Cost}_\alpha \). Also, the adverse selection component satisfies \( s^a = (g^s(\rho, \alpha) - g(\rho, \alpha)) \sigma_E = -g^a(\rho, w) \sigma_E = \text{Adverse Selection Cost}_\alpha \).

Verification of Result 2. See Internet Appendix 3.

Proof of Proposition 4. With the notation of Lemma A1 in this Appendix, consider a trader that perceives the signal \( w_t = \frac{v_t - v^f_t}{\sigma_E} \) distributed according to the normalized steady-state density prior, \( N(\mu_t, \sigma^2_t) = N(0, 1) \). Denote by \( \mu_{t+1} = f(\mu_t) \) the average posterior mean. Then, by setting \( \mu_t = 0 \) and \( \sigma_t = 1 \) in equation (A8), we obtain the desired formula for the resiliency coefficient \( K_0 \).

The remaining theoretical proofs and verification of numerical results is in the Internet Appendix.

References


